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The generating functional for hadronic weak interactions and its quenched approximation

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Abstract: We derive the generating functional of $|\Delta S| = 1, 2$ hadronic weak interactions at one loop for a generic number of flavours and its counterpart in the quenched approximation. A systematic analysis of the ultraviolet divergences in the full theory (with and without a singlet dynamical field) and in the quenched case is performed. We show that the quenched chiral logarithms in the presence of weak interactions amount to a redefinition of the weak mass term in the $\Delta S = \pm 1$ weak effective Lagrangian at leading order. Finally, we apply the results to B_K and $K \to \pi\pi$ matrix elements with $\Delta I = 1/2, 3/2$ to analyze the modifications induced by quenching on the coefficients of chiral logarithms in the one-loop corrections.

Keywords: Weak Decays, Kaon Physics, Lattice QCD, Chiral Lagrangians.

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Contents

1. Introduction

Nonleptonic weak interactions of light mesons are still one of the main unresolved issues of the Standard Model (SM). Two major aspects still await for a theoretical understanding: the $\Delta I = 1/2$ rule and the mechanism which mediates CP violation. Also, interaction mechanisms beyond the SM find a rich spectrum of implications in this context.

The short-distance QCD description of the operators that mediate nonleptonic weak transitions has been clearly formulated [1]. QCD asymptotic freedom and Operator Product Expansion are used to integrate out heavy degrees of freedom of the SM Lagrangian down to $\mu < m_c$. At this scale nonleptonic weak transitions are mediated by effective four-quark operators of light quarks u, d, s. The QCD evolution of their Wilson coefficients is determined by RG equations.

The evaluation of the matrix elements of the four-quark effective operators between light meson states is instead a pure non perturbative problem. It involves the knowledge of QCD contributions at long distances, namely from $\mu \sim m_c$ down to scales $\mu \simeq \Lambda_{QCD}$ or the typical light meson mass.

At long distances two approaches are viable. The first is the implementation of light hadrons matrix elements on the lattice, the second is the use of an effective Lagrangian for weak interactions where the dynamical degrees of freedom are the light mesons. The weak Lagrangian can be formulated on the same principles as the Chiral Perturbation Theory (ChPT) Lagrangian for strong interactions. It is an expansion in powers of the momenta of light mesons and light quark masses. The leading order weak Lagrangian is known since a long time [2, 3], while the derivation of the next-to-leading order Lagrangian was done in refs. [4, 5] for $N = 3$ flavours. This second approach is based on symmetry principles. Its advantage is that it allows for a perturbative treatment of weak processes at long distances. Its major drawback is the lack of predictivity in the parameters, residues of short-distance physics, which regulate the various interactions.

The computation of weak matrix elements on the lattice originates from *first principles* and in this sense it gives the correct answer to the problem. On the other hand it still suffers from the presence of major sources of systematic errors: finite volume effects, unphysical quark masses, the need of computing *unphysical* matrix elements [6] (see also ref. [7] for recent alternative proposals), in order to maximally avoid mixing with lower dimension operators in decay processes with two or more particles in the final state, and finally the fact that still most of the lattice evaluations are done in the quenched approximation. In this work we focus on formal aspects of the last problem within the ChPT framework, while in the immediate phenomenological applications we shall explore in a quantitative way the deviations of the quenched weak matrix elements from the physical ones. Further results on this last argument will be reported in ref. [8].

Our main scope is to provide a systematic treatment of the quenched approximation for nonleptonic weak interactions. On the other hand, the same approach will allow to add a few new results to previous analyses of weak interactions within the full (unquenched) theory. The approach that we adopt is the one of the generating functional within the framework of the low energy effective Lagrangian. This approach was applied in refs. [9, 10] to the quenched approximation of strong interactions (also known as quenched ChPT [11]). The first derivation of the generating functional of standard (unquenched) ChPT can be found in refs. [12, 13]. The analogous derivation in the weak sector can be found in refs. [4, 5], where the generating functional of hadronic weak interactions has been derived in the full theory with $N = 3$ flavours.

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In this work we calculate the ultraviolet divergences of the weak generating functional at one loop in the full theory and in the quenched approximation. This involves, as a first step, the derivation of the generating functional in the full theory with a generic number of flavours N and in the presence of a singlet dynamical field. The flavour number dependence was not known before, while the inclusion of the singlet dynamical field was sketched in ref. [5] (see Appendix A thereof) in the analysis of the $(8_L, 1_R)$ sector. Here we introduce the singlet dynamical field at first place and derive the modifications induced both in the $(8_L, 1_R)$ and the $(27_L, 1_R)$ operators which carry the ultraviolet divergences. As an outcome, we construct the minimal basis of ultraviolet divergent effective meson operators which appear at next-to-leading order both in the octet and 27-plet weak Lagrangians. We limit the present analysis to the case of degenerate light quark masses. In this case, once the ultraviolet divergences of the generating functional are known in the full and in the quenched theory, all the coefficients of chiral logarithms are known in both theories, for any nonleptonic weak matrix element of light mesons. For non degenerate light quark masses, also ultraviolet finite logarithms of the type $\log(m_K^2/m_\pi^2)$ do appear.

The whole derivation is done at infinite volume, with the intention of clarifying some of the formal aspects of the quenched approximation to weak interactions. The infrared behaviour of the quenched approximation in the presence of weak interactions is not considered here. However, we anticipate that its analysis with the use of the generating functional follows the same lines of the analysis done in the case of purely strong interactions in ref. [10].

The next question to answer is how much the quenched approximation modifies the predictions of weak matrix elements. Knowing the value of all the chiral logarithms in the full theory and the corresponding one in the quenched approximation, one can at least quantitatively predict how much quenching modifies their contribution to weak matrix elements. Here, we focus on the analysis of B_K and $K \to \pi\pi$ matrix elements. In the last case quenching effects turn out to be of considerable size. Also, their pattern in respect to the $\Delta I = 1/2$ rule can be clarified: while the full ChPT one loop corrections tend to support the enhancement of the $\Delta I = 1/2$ amplitude, quenched modifications (at infinite volume) tend to suppress the $\Delta I = 1/2$ dominance. The modifications induced by quenching, together with the analysis of *unphysical* $K \to \pi\pi$ matrix elements as they are implemented on the lattice are further considered in ref. [8].

The plan of the paper is as follows. In section 2 the effective Lagrangian for hadronic weak interactions with $|\Delta S| = 1$, 2 is extended to its graded version, that reproduces the quenched approximation within Chiral Perturbation Theory. In section 3 the ultraviolet divergences of the generating functional for hadronic weak interactions at one loop are derived in the full theory for a generic number of flavours and in the quenched approximation. A peculiar behaviour of the quenched approximation in the presence of weak interactions is the generation of quenched chiral logarithms in addition to the ones produced by strong interactions [10]. How they can be formally interpreted as the rescaling of a parameter in the leading order weak Lagrangian is clarified in section 3.3. We treat separately the bosonic contribution to the generating functional in subsections 3.1, 3.2, 3.3 and 3.4, and the fermionic contribution in subsection 3.5. The reader who is not interested in the technical details of the derivation can skip the full section 3. In section 4 we give the final result for the ultraviolet divergences in the quenched approximation, while in subsection 4.1 we define the divergent counterterms in the full theory and compare with their quenched counterpart (see tables 1 and 2 for the comparison). This analysis is basically useful for the phenomenological applications considered in section 5. Here we focus on the contribution to weak observables coming from the chiral logarithms and how they are modified by quenching. The analysis is performed at infinite volume. In particular, in subsection 5.1 we rederive known results for the B_K parameter, with the aim of clarifying the structure of the counterterms and their flavour number dependence. In subsection 5.2 we analyze $K \to \pi\pi$ matrix elements with $\Delta I = 1/2$ and $3/2$; we consider the full matrix elements and their quenched approximation. Since we work in the continuum at infinite volume, we give all the predictions in Minkowski space-time. We conclude in section 6. There are two appendices. In Appendix A the list of divergent counterterms in the octet and 27-plet case is given and in the presence of the singlet field. In Appendix B the single contributions to the bosonic determinant in the singlet sector are illustrated.

2. The quenched Lagrangian for weak interactions

The complete quenched ChPT Lagrangian for light pseudoscalar mesons at leading order in the chiral expansion (i.e. at order p^2 and linear in the light quark masses) can be written as follows:

$$\mathcal{L} = \mathcal{L}_{\Delta S=0} + \mathcal{L}_{\Delta S=1} + \mathcal{L}_{\Delta S=2} \,. \tag{2.1}$$

The Lagrangian $\mathcal{L}_{\Delta S=0}$ is the quenched version of the leading order Lagrangian for strong interactions [10, 11]

$$\begin{aligned}\mathcal{L}_{\Delta S=0} = & V_1(\Phi_0)\mathrm{str}(u_{\mu s}u_s^{\mu}) + V_2(\Phi_0)\mathrm{str}(\chi_{+s}) - V_0(\Phi_0) + \\ & + V_5(\Phi_0)D_\mu\Phi_0 D^\mu\Phi_0 \,.\end{aligned} \tag{2.2}$$

It is invariant under the graded chiral symmetry group $[SU(N|N)_L \otimes SU(N|N)_R] \odot U(1)_V$ with N physical flavours and N ghost flavours. The graded fields[1] are defined with the usual notation [10]

$$\begin{aligned}u_{\mu s} &= iu_s^\dagger D_\mu U_s u_s^\dagger = u_{\mu s}^\dagger \,, \\ \chi_{+s} &= u_s^\dagger \chi_s u_s^\dagger + u_s \chi_s^\dagger u_s \,,\end{aligned} \tag{2.3}$$

[1] The quenched counterpart of a standard CHPT quantity is either denoted with capital letters (as in $\phi \to \Phi$) or with the s subscript

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where $U_s = u_s^2$ is the exponential representation of the graded meson field:

$$U_s = \exp\left(\sqrt{2}i\,\Phi/F\right),$$

F is the bare quenched pion decay constant (with renormalized value $F_\pi = 93$ MeV) and Φ is a hermitian non traceless 2×2 block matrix

$$\Phi = \begin{pmatrix} \phi & \theta^\dagger \\ \theta & \tilde{\phi} \end{pmatrix}, \qquad \mathrm{str}(\Phi) = \Phi_0 = \phi_0 - \tilde{\phi}_0 .$$

The potentials $V_i(\Phi_0)$ in eq. (2.2) are real and even functions of the super-η' field $\Phi_0 = \mathrm{str}(\Phi)$.

The Lagrangians $\mathcal{L}_{\Delta S=1}$ and $\mathcal{L}_{\Delta S=2}$ in eq. (2.1) are the quenched version of the leading order Lagrangians which mediate strangeness changing non leptonic weak interactions in one and two units. The quenched version of the weak effective Lagrangians is the generalization to a graded group (i.e. with N physical flavours and N ghost flavours) of the leading order weak effective Lagrangians given in refs. [3, 4, 5], with the inclusion of a singlet dynamical field. Using the same notation as for the strong Lagrangian, they can be written as follows:

$$\begin{aligned}\mathcal{L}_{\Delta S=1} = {} & \tilde{V}_8(\Phi_0)\mathrm{str}\left(\Delta_{s32}u_{\mu s}u_s^\mu\right) + \tilde{V}_5(\Phi_0)\mathrm{str}\left(\Delta_{s32}\chi_{+s}\right) + \\ & +\tilde{V}_0(\Phi_0)\mathrm{str}\left(\Delta_{s32}u_{\mu s}\right)\mathrm{str}(u_s^\mu) + \mathrm{h.c} + \\ & +\tilde{V}_{27}(\Phi_0)t^{ij,kl}\mathrm{str}\left(\Delta_{sij}u_{\mu s}\right)\mathrm{str}\left(\Delta_{skl}u_s^\mu\right)\end{aligned} \tag{2.4}$$

and

$$\mathcal{L}_{\Delta S=2} = \tilde{V}_{27}^{\Delta S=2}(\Phi_0)\, t^{ij,kl}\mathrm{str}(\Delta_{sij}u_{\mu s})\mathrm{str}(\Delta_{skl}u_s^\mu)\,, \tag{2.5}$$

where the potentials $\tilde{V}_i(\Phi_0)$ are again real and even functions of the super-η' field Φ_0. The $\Delta S = 1$ Lagrangian in eq. (2.4) contains the octet $(8_L, 1_R)$ operators in the first line and the 27-plet $(27_L, 1_R)$ operator in the second line. The octet term associated to the potential $\tilde{V}_0(\Phi_0)$ is induced by the inclusion of the singlet dynamical field.

The operator Δ_{sij} is the graded realization of the usual projection matrix onto the octet components of the chiral fields. It is given by $\Delta_{sij} = u_s\lambda_{ij}\frac{1+\tau_3}{2}u_s^\dagger$, with $(\lambda_{ij})_{ab} = \delta_{ia}\delta_{jb}$. The tensor $t^{ij,kl}$ in eqs. (2.4), (2.5) projects onto the $(27_L, 1_R)$ component of the interacting fields, with $|\Delta S| = 1$ or 2. It is a completely symmetric tensor that satisfies

$$t^{ij,kl} = t^{kl,ij} \tag{2.6}$$

and the following trace zero conditions

$$\sum_i t^{ii,kl} = 0\,, \qquad \sum_i t^{ij,ki} = 0\,. \tag{2.7}$$

Note that in the fundamental (unquenched) theory the first condition in eq. (2.7) guarantees that only the traceless component of any operators $\mathcal{O}_1$, $\mathcal{O}_2$ contributes to the $(27_L, 1_R)$ term $t^{ij,kl}\mathrm{tr}(\lambda_{ij}\mathcal{O}_1)\mathrm{tr}(\lambda_{kl}\mathcal{O}_2)$, with $(\lambda_{ij})_{ab} = \delta_{ia}\delta_{jb}$.

In the quenched case the same condition guarantees that only the traceless part of the $(1,1)$ component of the graded operators $\mathcal{O}_{s1}$, $\mathcal{O}_{s2}$ contributes to the $(27_L, 1_R)$ term $t^{ij,kl}\mathrm{str}(\lambda_{ij}\frac{1+\tau_3}{2}\mathcal{O}_{s1})\mathrm{str}(\lambda_{kl}\frac{1+\tau_3}{2}\mathcal{O}_{s2})$.

In the $\Delta S = \pm 1$ case and in $SU(3)$ the $t^{ij,kl}$ tensor has the following components

$$\begin{aligned} t^{21,13} &= t^{13,21} = t^{12,31} = t^{31,12} = \frac{1}{3}\,, \\ t^{22,23} &= t^{23,22} = t^{22,32} = t^{32,22} = -\frac{1}{6}\,, \\ t^{33,23} &= t^{23,33} = t^{33,32} = t^{32,33} = -\frac{1}{6}\,, \\ t^{11,23} &= t^{23,11} = t^{11,32} = t^{32,11} = \frac{1}{3} \end{aligned} \tag{2.8}$$

and 0 otherwise, while for $\Delta S = 2$ interactions it is given by[2]

$$t^{23,23} = t^{32,32} = 1\,, \qquad 0 \text{ otherwise}\,. \tag{2.9}$$

The weak Lagrangians in eqs. (2.4) and (2.5) are CPS invariant [3], CP even and S even, where the S ("switching") invariance denotes the invariance under the interchange of 2 and 3 components ($s \leftrightarrow d$). One can simply obtain the CP odd, S odd $\Delta S = 1$ Lagrangian from eq. (2.4) by replacing everywhere $\tilde{V}_i \to i\tilde{V}_i^-$, $\Delta_{s32} \to -\Delta_{s32}$ in the octet terms and assigning the opposite sign to $t^{32,ii}$, $t^{ii,32}$, $t^{21,13}$, $t^{13,21}$ in the 27-plet terms.

The symmetry under the graded group is made local in the presence of graded external sources $l_s^\mu, r_s^\mu, s_s, p_s$. The covariant derivative over the field U_s is defined as $D^\mu U_s = \partial^\mu U_s - ir_s^\mu U_s + iU_s l_s^\mu$ and the field $\chi_s = 2B_0(s_s + ip_s)$ of eq. (2.3) contains the external scalar (s_s) and pseudoscalar (p_s) sources. Since we are not interested in studying matrix elements with spurious fields as external legs we put to zero all spurious external sources. With this reduction a generic graded source reads

$$j_s = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \qquad j = p, v_\mu, a_\mu\,,$$

and the scalar source contains the quark mass matrix $\mathcal{M}$ at the leading order

$$s_s = \begin{pmatrix} \mathcal{M} + \delta s & 0 \\ 0 & \mathcal{M} \end{pmatrix}.$$

In what follows the quark mass matrix will be taken proportional to the unit matrix: $\mathcal{M} = m_q\mathbf{1}$.

[2] In the generalization to $N > 3$ flavours the same tensor as in eqs. (2.8) and (2.9) is used.

All the potentials $V_i(\Phi_0)$ and $\tilde{V}_i(\Phi_0)$ in eqs. (2.2, 2.4, 2.5) can be expanded in powers of Φ_0^2. We define the first terms in the expansion as follows

$$
\begin{aligned}
V_0(\Phi_0) &= \frac{m_0^2}{2N_c}\Phi_0^2 + O\left(\Phi_0^4\right) , \\
V_{1,2}(\Phi_0) &= \frac{F^2}{4} + \frac{1}{2}v_{1,2}\,\Phi_0^2 + O\left(\Phi_0^4\right) , \\
V_5(\Phi_0) &= \frac{\alpha}{2N_c} + O\left(\Phi_0^2\right) ,
\end{aligned}
\tag{2.10}
$$

in the strong sector, where m_0^2 is the squared singlet mass and α is a new parameter associated with the kinetic term of the singlet field,

$$
\begin{aligned}
\tilde{V}_8(\Phi_0) &= g_8 + \frac{1}{2}\tilde{v}_8\,\Phi_0^2 + O\left(\Phi_0^4\right) , \\
\tilde{V}_5(\Phi_0) &= g_8' + \frac{1}{2}\tilde{v}_5\,\Phi_0^2 + O\left(\Phi_0^4\right) , \\
\tilde{V}_{27}(\Phi_0) &= g_{27} + \frac{1}{2}\tilde{v}_{27}\,\Phi_0^2 + O\left(\Phi_0^4\right) , \\
\tilde{V}_0(\Phi_0) &= \overline{g}_8 + \frac{1}{2}\tilde{v}_0\,\Phi_0^2 + O\left(\Phi_0^4\right)
\end{aligned}
\tag{2.11}
$$

in the $|\Delta S| = 1$ weak sector and

$$
\tilde{V}_{27}^{\Delta S=2}(\Phi_0) = g_{27}^{\Delta S=2} + \frac{1}{2}\tilde{v}_{27}^{\Delta S=2}\,\Phi_0^2 + O\left(\Phi_0^4\right) \tag{2.12}
$$

in the $\Delta S = 2$ weak sector. The weak couplings g_8, g_{27}, g_8', $\bar{g}_8$ and $g_{27}^{\Delta S=2}$ can be written in terms of dimensionless couplings as follows

$$
\begin{aligned}
g_8 &= CF^4G_8\,, \qquad g_{27} = CF^4G_{27}\,, \qquad g_8' = CF^4G_8'\,, \\
\bar{g}_8 &= CF^4\bar{G}_8\,, \qquad g_{27}^{\Delta S=2} = C^{\Delta S=2}F^4G_{27}\,,
\end{aligned}
\tag{2.13}
$$

where the constant C contains the CKM matrix elements via

$$
C = -\frac{3}{5}\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\,, \tag{2.14}
$$

while the constant $C^{\Delta S=2}$ can be written as follows

$$
C^{\Delta S=2} = -\frac{G_F}{4}\mathcal{F}\left(m_t^2,\, m_c^2,\, M_W^2,\, \mathbf{V}_{CKM}\right) , \tag{2.15}
$$

where $\mathcal{F}(m_t^2,\, m_c^2,\, M_W^2,\, V_{CKM})$ is a known function of the W boson mass, the heavy quark masses and CKM matrix elements [14]. In the large-N_c limit the dimensionless couplings G_8 and G_{27} are $G_8 = G_{27} = 1$. Notice that the 27-plet couplings g_{27} and $g_{27}^{\Delta S=2}$ are the same, modulo the weak-sector coefficients C and $C^{\Delta S=2}$. We expect that the same is true for the full potentials $\tilde{V}_{27}$ and $\tilde{V}_{27}^{\Delta S=2}$, so that $\tilde{V}_{27}^{\Delta S=2} = (C^{\Delta S=2}/C)\,\tilde{V}_{27}$.

3. The Weak generating functional to one loop

The derivation of the generating functional for hadronic weak interactions and its counterpart in the quenched approximation can be done following the same lines of ref. [10], where the quenched generating functional for low energy strong interactions has been derived within the framework of ChPT. We expand the leading order action around the classical solution up to and including quadratic fluctuations. We write the field U_s as:

$$U_s = u_s \; e^{i\Xi} \; u_s \, ,$$

where $\bar{U_s} = u_s^2$ is the classical solution to the equations of motion. In the absence of spurious external sources it reduces to

$$u_s = \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix} .$$

We decompose the fluctuation Ξ similarly to the field Φ and write:

$$\Xi = \begin{pmatrix} \xi & \zeta^\dagger \\ \zeta & \tilde{\xi} \end{pmatrix}, \qquad \mathrm{str}(\Xi) = \sqrt{N}(\xi_0 - \tilde{\xi}_0) \, .$$

Each field can be decomposed in terms of the generalized $\hat{\lambda}_a$ matrices

$$\xi = \sum_{a=0}^{N^2-1} \hat{\lambda}_a \xi^a \, , \tag{3.1}$$

where $\hat{\lambda}_a = \lambda_a/\sqrt{2}, \mathbf{1}/\sqrt{N}$ for $a = 1, \ldots N^2 - 1$ and $a = 0$ respectively. It is useful to define a graded vector for the bosonic fields as follows: $X^T = (\xi^T, \tilde{\xi}^T)$ and $\xi^T = (\xi^0, \xi^1, \xi^2, \ldots, \xi^{N^2-1})$, the same for $\tilde{\xi}^T$. The fermionic ghost fields are collected in the vector $\zeta^\dagger = (\zeta^{\dagger 0}, \zeta^{\dagger 1}, \ldots, \zeta^{\dagger N^2-1})$. With this notation the complete action for strong and weak interactions up to quadratic fluctuations can be written in a compact form[3]

$$S[\Phi] = S[\bar{\Phi}] - \frac{F^2}{4} \int \, dx \left\{ X^T D X + 2\zeta^\dagger D_\zeta \zeta \right\} + O(\Xi^3) \, , \tag{3.2}$$

where summation over flavour indices is implicit. Notice also that the bosonic differential operator D is a graded 2×2 matrix acting on X. Using the action (3.2), the quenched generating functional to one loop can be formally written as follows

$$e^{iZ^{\mathrm{q}}_{1\,\mathrm{loop}}} = \mathcal{N} \frac{\det D_\zeta}{(\det D)^{\frac{1}{2}}} \, . \tag{3.3}$$

In the following subsections we treat separately the bosonic part and the fermionic ghost part of the generating functional in eq. (3.3).

[3] As we did in the strong case we disregard the infinite chain of terms containing powers of the super-η' field at the classical solution $\bar{\Phi}_0$, coming from the expansion of the potentials $V_i(\Phi_0)$ and $\tilde{V}_i(\Phi_0)$.

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3.1 The bosonic sector

While in the purely strong case treated in ref. [10] a mixing is only induced amongst the singlet components of the physical ξ and the ghost $\tilde{\xi}$ fluctuation field, weak interactions induce a mixing also in the non singlet sector. For this reason a compact representation of the generating functional in the bosonic sector in terms of graded vectors and graded operators turns out to be convenient. The differential operator D in eq. (3.2) is a 2×2 graded differential matrix acting on the graded vector X. It can be written as follows

$$D_{ab} = d_{\mu s} G_{ab} d_s^{\mu} + F_{ab} \,, \tag{3.4}$$

where d_s^{μ} is the graded covariant derivative given by

$$d_{\mu s} = d_{\mu} \frac{1+\tau_3}{2} + \tilde{d}_{\mu} \frac{1-\tau_3}{2} \,, \tag{3.5}$$

with d_{μ} the covariant derivative acting on ξ and $\tilde{d}_{\mu}$ the covariant derivative acting on $\tilde{\xi}$. In the absence of spurious external sources $\tilde{v}_{\mu}, \tilde{a}_{\mu} = 0$ the latter reduces to the partial derivative $\tilde{d}_{\mu} = \partial_{\mu}$. The graded operator G_{ab} is given by

$$G_{ab} = \delta_{ab} \tau_3 + \alpha_{ab} \tag{3.6}$$

and it is hermitian. The operators F_{ab} in eq. (3.4) and α_{ab} in eq. (3.6) are in turn 2×2 graded matrices. The term α_{ab} is induced by weak interactions and we call it the weak connection in analogy with the connection tensor which appears in the metric of a curved space. To compute the ultraviolet divergences of the generating functional to one loop with standard techniques it is appropriate to rescale the coefficient of the d'Alambertian in eq. (3.4) to one. We generalize the rescaling procedure used in ref. [4] to the quenched case and in the presence of a singlet component. We define the hermitian operator G as $G = gg = \bar{g}^T \tau_3 \bar{g}$, where $\bar{g}_{ab} = \sqrt{\delta_{ab} + \tau_3 \alpha_{ab}}$ and $\bar{g}^T_{ab} = \sqrt{\delta_{ab} + \alpha_{ab} \tau_3}$. By commuting g with the covariant derivative in eq. (3.4) it is easy to show that the following identity holds

$$d_s g g d_s = g d_s d_s g + [d_s, g] d_s g - g d_s [d_s, g] - [d_s, g][d_s, g] \,. \tag{3.7}$$

With the use of the $\bar{g}$ definition the quadratic fluctuation of the bosonic Lagrangian can be rewritten as $X'^T D' X'$, where the rescaled differential operator is now given by

$$D' = d_s^2 \tau_3 + \bar{g}^{T^{-1}} F \, \bar{g}^{-1} + \bar{g}^{T^{-1}} \left([d_s, g] d_s g - g d_s [d_s, g] - [d_s, g][d_s, g] \right) \bar{g}^{-1} \,, \tag{3.8}$$

where flavour indices have been omitted for simplicity, and the rescaled fluctuation field is defined as $X' = \bar{g} X, X'^T = X^T \bar{g}^T$. This is valid provided that $\bar{g}^{-1}$ does exist, that is our case. The rescaled operator D' has now the treatable form with unit coefficient in the double derivative term. In terms of the rescaled fluctuation fields the bosonic part of the generating functional is now given by

$$e^{iZ_b} = \mathcal{N} e^{i\overline{Z}_b} \int dX'^{T} dX' \, (\det G\tau_3)^{-\frac{1}{2}} \; e^{-i\frac{F^2}{4}\int dx\, X'^{T} D' X'} \,, \qquad (3.9)$$

where $\overline{Z}_b$ is the classical contribution to the generating functional and we made use of the identity $\det(\bar{g}^T\bar{g}) = \det(gg\tau_3) = \det(G\tau_3)$. The complete bosonic determinant is now given by $(\det D)^{-1/2} = (\det D')^{-1/2}(\det G\tau_3)^{-1/2}$.

As it is the relevant one, we limit the analysis of the generating functional to the first order in the expansion in powers of the weak interaction coupling G_F. The weak connection α_{ab} in eq. (3.6) is of order G_F, while the operator F_{ab} contains both strong interaction terms and order G_F ones. Its explicit expression can be written as follows:

$$F_{ab} = -\frac{1}{2}\left\{N^-_{ab}, d_s\right\} + \frac{1}{2}\left[d_s, N^+_{ab}\right] + \hat{\sigma}_{ab} + \hat{\omega}_{ab} \,, \qquad (3.10)$$

where $\hat{\sigma}_{ab}$ is the usual graded strong interaction operator as defined in ref. [10], while N^-_{ab}, N^+_{ab} and $\hat{\omega}_{ab}$ are graded weak interaction operators.

Expanding g and $\bar{g}^{-1}$ in powers of the weak connection α_{ab} (i.e. in powers of G_F) in eq. (3.8) and keeping up to order G_F terms, the rescaled bosonic differential operator can be written as follows

$$D'_{ab} = d_s^2\delta_{ab}\tau_3 + F_{ab} - \frac{1}{2}\left(\alpha_{al}\tau_3\hat{\sigma}_{lb} + \hat{\sigma}_{al}\tau_3\alpha_{lb}\right) - \frac{1}{2}\left[d_s, [d_s, \alpha_{ab}]\right] + O\left(G_F^2\right) \,, \qquad (3.11)$$

with F_{ab} given in eq. (3.10).

The operators F_{ab} and α_{ab} act on various subspaces of the graded space. They can be classified as follows: (a) the physical subspace of the fluctuation field ξ with projection operator $(1+\tau_3)/2$, (b) the bosonic ghost subspace of the fluctuation field $\tilde{\xi}$ with projection operator $(1-\tau_3)/2$, (c) the mixed physical–bosonic ghost subspace with projection operator $(1-\tau_1)$, or $[(1+\tau_3)/2 - (\tau_1 + i\tau_2)/2]$ and its transposed. For this reason the weak connection α_{ab} admits the following decomposition into the four projected components:

$$\alpha_{ab} = \alpha^{11}_{ab}\frac{1+\tau_3}{2} + \alpha^{22}_{ab}\frac{1-\tau_3}{2} + \alpha^{12}_{ab}\frac{\tau_1 + i\tau_2}{2} + \alpha^{21}_{ab}\frac{\tau_1 - i\tau_2}{2} \,, \qquad (3.12)$$

where[4]

$$\begin{aligned}
\alpha^{11}_{ab} &= \frac{1}{2}g_8 k \left\langle \Delta\{\hat{\lambda}_a, \hat{\lambda}_b\}\right\rangle + g_{27} q \left\langle \Delta\hat{\lambda}_a\right\rangle \left\langle \Delta\hat{\lambda}_b\right\rangle + \frac{1}{2}\bar{g}_8\sqrt{N}k\left(\left\langle \Delta\hat{\lambda}_a\right\rangle\delta_{0b} + \left\langle\Delta\hat{\lambda}_b\right\rangle\delta_{0a}\right) \\
\alpha^{22}_{ab} &= 0 \\
\alpha^{12}_{ab} &= -\frac{1}{2}\bar{g}_8\sqrt{N}k\left\langle\Delta\hat{\lambda}_a\right\rangle\delta_{0b} \\
\alpha^{21}_{ab} &= -\frac{1}{2}\bar{g}_8\sqrt{N}k\left\langle\Delta\hat{\lambda}_b\right\rangle\delta_{0a} \,.
\end{aligned} \qquad (3.13)$$

[4] The notation $\langle \ldots \rangle$ stands everywhere for the trace over flavour indices.

Here and in the following we use a short-hand notation for the octet and 27-plet operators as follows:

$$k \cdot \Delta \equiv \frac{4}{F^2} f^{ij} \Delta_{ij} \qquad f^{23} = f^{32} = 1\,, \qquad 0 \text{ otherwise} \tag{3.14}$$

for the octet case and

$$q \cdot \Delta \cdot \Delta \equiv \frac{4}{F^2} t^{ij,kl} \Delta_{ij} \Delta_{kl} \tag{3.15}$$

in the 27-plet case. The tensor $t^{ij,kl}$ has been defined in eq. (2.8) for $\Delta S = \pm 1$ and in eq. (2.9) for $\Delta S = 2$.

The operators N_μ^- and N_μ^+ admit the same decomposition as in eq. (3.12). It gives

$$\begin{aligned}
N^{-11}_{\mu\, ab} &= \frac{i}{4} g_8 k \left\langle \{\Delta, u_\mu\} \left[\hat\lambda_a, \hat\lambda_b\right]\right\rangle + \frac{i}{2} g_8 k \left\langle \Delta \left(\hat\lambda_a u_\mu \hat\lambda_b - \hat\lambda_b u_\mu \hat\lambda_a\right)\right\rangle + \\
&\quad + i g_{27} q \left\langle \Delta \left[\hat\lambda_a, \hat\lambda_b\right]\right\rangle \langle \Delta u_\mu \rangle - \frac{i}{2} g_{27} q \left\langle [\Delta, u_\mu] \hat\lambda_a \right\rangle \left\langle \Delta \hat\lambda_b \right\rangle + \\
&\quad + \frac{i}{2} g_{27} q \left\langle [\Delta, u_\mu] \hat\lambda_b \right\rangle \left\langle \Delta \hat\lambda_a \right\rangle + \frac{i}{2} \bar g_8 k \left\langle \Delta \left[\hat\lambda_a, \hat\lambda_b\right]\right\rangle \langle u_\mu \rangle - \\
&\quad - \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_a\right]\right\rangle \delta_{0b} + \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_b\right]\right\rangle \delta_{0a}\,, \\
N^{-22}_{\mu\, ab} &= 0\,, \\
N^{-12}_{\mu\, ab} &= \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_a\right]\right\rangle \delta_{0b}\,, \\
N^{-21}_{\mu\, ab} &= -\frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_b\right]\right\rangle \delta_{0a}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
N^{+11}_{\mu\, ab} &= -\frac{i}{4} g_8 k \left\langle [\Delta, u_\mu] \left\{\hat\lambda_a, \hat\lambda_b\right\}\right\rangle - \frac{i}{2} g_{27} q \left\langle [\Delta, u_\mu] \hat\lambda_a \right\rangle \left\langle \Delta \hat\lambda_b \right\rangle - \\
&\quad - \frac{i}{2} g_{27} q \left\langle [\Delta, u_\mu] \hat\lambda_b \right\rangle \left\langle \Delta \hat\lambda_a \right\rangle - \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_a\right]\right\rangle \delta_{0b} - \\
&\quad - \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_b\right]\right\rangle \delta_{0a}\,, \\
N^{+22}_{\mu\, ab} &= 0\,, \\
N^{+12}_{\mu\, ab} &= \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_a\right]\right\rangle \delta_{0b}\,, \\
N^{+21}_{\mu\, ab} &= \frac{i}{4} \bar g_8 \sqrt{N} k \left\langle \Delta \left[u_\mu, \hat\lambda_b\right]\right\rangle \delta_{0a}\,.
\end{aligned} \tag{3.17}$$

The weak operator $\hat\omega_{ab}$ and the strong operator $\hat\sigma_{ab}$ can be decomposed in a more convenient manner as follows

$$\hat\Theta_{ab} = \hat\Theta^{11}_{ab} \frac{1+\tau_3}{2} + \hat\Theta^{22}_{ab} \frac{1-\tau_3}{2} + \Delta\hat\Theta \delta_{a0} \delta_{b0} (1 - \tau_1)\,, \tag{3.18}$$

where the last term is a pure singlet contribution that mixes bosonic ghost and physical singlet fields. The components of the weak operator $\hat{\omega}_{ab}$ are

$$
\begin{aligned}
\hat{\omega}^{11}_{ab} &= \frac{1}{8} g_8 k \left(\left\langle \{\Delta, u^2\} \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\}\right\rangle - \left\langle \Delta \left(\hat{\lambda}_a u^2 \hat{\lambda}_b + \hat{\lambda}_b u^2 \hat{\lambda}_a\right)\right\rangle - \right. \\
&\qquad - \left\langle \{\Delta, u_\mu\} \left(\hat{\lambda}_a u^\mu \hat{\lambda}_b + \hat{\lambda}_b u^\mu \lambda_a\right)\right\rangle + \left. \left\langle \Delta u_\mu \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\} u^\mu \right\rangle\right) + \\
&\quad + \frac{1}{8} g_8' k \left(\left\langle \{\Delta, \chi_+\} \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\}\right\rangle - \left\langle [\Delta, \chi_-] \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\}\right\rangle\right) + \\
&\quad + \frac{1}{4} g_{27} q \left(\langle \Delta u_\mu \rangle \left\langle \{\Delta, u_\mu\} \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\}\right\rangle - 2 \langle \Delta u_\mu \rangle \left\langle \Delta \left(\hat{\lambda}_a u^\mu \hat{\lambda}_b + \hat{\lambda}_b u^\mu \lambda_a\right)\right\rangle + \right. \\
&\qquad \left. + \left\langle [\Delta, u_\mu] \hat{\lambda}_a \right\rangle \left\langle [\Delta, u^\mu] \hat{\lambda}_b \right\rangle\right) - \\
&\quad - \frac{1}{4} \bar{g}_8 \langle u_\mu \rangle k \left\langle \Delta \left(\hat{\lambda}_a u^\mu \hat{\lambda}_b + \hat{\lambda}_b u^\mu \hat{\lambda}_a - \frac{1}{2} \left\{u^\mu, \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\}\right\}\right)\right\rangle , \\
\hat{\omega}^{22}_{ab} &= 0, \\
\Delta \hat{\omega} &= -N \left(\tilde{v}_0 k \langle \Delta u^\mu \rangle \langle u_\mu \rangle + \tilde{v}_8 k \langle \Delta u^2 \rangle + \tilde{v}_5 k \langle \Delta \chi_+ \rangle + \tilde{v}_{27} q \langle \Delta u_\mu \rangle \langle \Delta u^\mu \rangle \right) , \qquad (3.19)
\end{aligned}
$$

while the strong operator $\hat{\sigma}_{ab}$ is given by

$$
\begin{aligned}
\hat{\sigma}^{11}_{ab} &= -\frac{1}{4} \left\langle \left[u_\mu, \hat{\lambda}_a\right] \left[u^\mu, \hat{\lambda}_b\right]\right\rangle + \frac{1}{4} \left\langle \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\} \chi_+ \right\rangle , \\
\hat{\sigma}^{22}_{ab} &= -\frac{1}{4} \left\langle \left\{\hat{\lambda}_a, \hat{\lambda}_b\right\} 4 B_0 \mathcal{M} \right\rangle , \\
\Delta \hat{\sigma} &= \frac{N}{3} \left(\alpha \Box + m_0^2\right) - N \left(v_1 \langle u_\mu u^\mu \rangle + v_2 \langle \hat{\chi}_+ \rangle\right) , \qquad (3.20)
\end{aligned}
$$

where we have defined the subtracted operator $\hat{\chi}_+ = \chi_+ - 4B_0\mathcal{M}$, which gives $\langle \hat{\chi}_+ \rangle = \langle \chi_+ \rangle - 2NM^2$, with M the bare meson mass, for degenerate quark masses.

In order to compute the ultraviolet divergences of the bosonic generating functional at one loop given in eq. (3.9), the next step to perform is to diagonalize the quadratic form $X'^T D' X'$ by eliminating the mixing between the singlet and non singlet components. We proceed in the same manner as it was done in ref. [10] for the purely strong sector. We decompose the quadratic form in its non singlet, mixed and singlet sectors as follows

$$
X'^T D' X' = \overline{X}^T \, \overline{D} \, \overline{X} + X_0^T \, B^T \, \overline{X} + \overline{X}^T \, B \, X_0 + X_0^T \, D_X \, X_0 \, , \qquad (3.21)
$$

where $\overline{D}$ acts on the non singlet graded vector $\overline{X}$, B and B^T are mixing operators between the non singlet $\overline{X}$ and singlet X_0 vectors and D_X acts on the singlet graded vector X_0. The mixing operator B_{a0} and its transposed B_{0a} are easily derived from the previous expressions, which give at order G_F

$$
B_{a0} = -\frac{1}{2} \left\{N^-_{a0}, d_s\right\} + \frac{1}{2} \left[d_s, N^+_{a0}\right] + \hat{\sigma}_{a0} + \hat{\omega}_{a0} - \frac{1}{2} \left(\alpha_{al} \tau_3 \hat{\sigma}_{l0} + \hat{\sigma}_{al} \tau_3 \alpha_{l0}\right) - \frac{1}{2} \left[d_s, \left[d_s, \alpha_{a0}\right]\right] . \qquad (3.22)
$$

By performing the translation

$$\overline{X} = \overline{X}' - \overline{D}^{-1} B X_0 \,, \tag{3.23}$$

the quadratic form in eq. (3.21) is diagonalized to

$$X'^T D' X' = \overline{X}'^T \overline{D}\,\overline{X}' + X_0^T \left(D_X - B^T \bar{D}^{-1} B \right) X_0 \,. \tag{3.24}$$

The first term on the r.h.s. is now a pure non singlet form, while in the second term the singlet operator D_X is shifted via a non local differential term. As already discussed in the case of strong interactions the non locality of the singlet differential operator does not prevent the derivation of the ultraviolet divergences of the one loop generating functional in closed form, since they remain local. Denoting with $\overline{D}_X = D_X - B^T \bar{D}^{-1} B$ the shifted singlet operator, the complete quenched generating functional to one loop can now be formally written as follows

$$e^{iZ^{\mathrm{q}}_{1\,\mathrm{loop}}} = \mathcal{N} \frac{\det D_\zeta}{(\det G\tau_3)^{\frac{1}{2}} (\det \bar{D})^{\frac{1}{2}} (\det \bar{D}_X)^{\frac{1}{2}}} \,, \tag{3.25}$$

where in the derivation of the ultraviolet divergences we retain up to order G_F terms in the weak sector. The bosonic contribution to the generating functional has now been splitted in its singlet and non singlet part.

3.2 Integral over the non-singlet fields

The operator $\bar{D}_{ab}$ acts as follows ($a, b = 1, \ldots N^2 - 1$):

$$\begin{aligned} \bar{D}_{ab} \bar{X}^b &= d_{s\mu} d_s^\mu \tau_3 \bar{X}_a + \left(\frac{1}{2} [d_s, N^+]_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab} - \frac{1}{2} \left(\alpha_{al} \tau_3 \hat{\sigma}_{lb} + \hat{\sigma}_{al} \tau_3 \alpha_{lb} \right) \right. \\ &\qquad\qquad \left. - \frac{1}{2} \left[d_s, [d_s, \alpha_{ab}] \right] \right) \bar{X}^b + O\left(G_F^2\right) \\ d_{s\mu} \bar{X}_a &= \partial_\mu \bar{X}_a + \hat{\Gamma}_{\mu ab} \bar{X}^b \quad , \end{aligned} \tag{3.26}$$

where we have written $\bar{D}_{ab}$ in terms of a rescaled covariant derivative which now contains the weak operator N^-_{ab}; the connection $\hat{\Gamma}_{\mu ab}$ is given by

$$\hat{\Gamma}_{\mu ab} = - \left\langle \Gamma_\mu \left[\hat{\lambda}_a, \hat{\lambda}_b \right] \right\rangle \frac{1+\tau_3}{2} - \frac{1}{2} N^-_{\mu ab} \,, \tag{3.27}$$

containing the usual strong interaction connection (first term), where

$$\Gamma_\mu = \frac{1}{2\left([u^\dagger, \partial_\mu u] - i u^\dagger r_\mu u - i u l_\mu u^\dagger \right)} \,,$$

and a weak contribution (second term). The projections of the graded operators $N^\pm_\mu, \alpha, \hat{\sigma}, \hat{\omega}$ onto the non-singlet subspace are easily derived from eqs. (3.13), (3.16), (3.17), (3.19) and (3.20).

By explicitly writing $\bar{D}_{ab}$ in terms of its projections onto the physical and non physical subspaces one realizes that a mixing between the physical ξ and non physical $\tilde{\xi}$ fluctuations is still induced by the weak connection α_{ab}. We get

$$\bar{X}^T \bar{D} \bar{X} = \xi^T D_\xi \xi - \tilde{\xi}^T \left(\Box - \hat{\sigma}^{22}\right) \tilde{\xi} + \xi^T \left(\frac{1}{2}\alpha^{12}\hat{\sigma}^{22}\right) \tilde{\xi} + \tilde{\xi}^T \left(\frac{1}{2}\hat{\sigma}^{22}\alpha^{21}\right) \xi \,, \tag{3.28}$$

where the first term on the r.h.s. contains the differential operator D_ξ which acts on the physical non-singlet field ξ_a, $a = 1, \ldots N^2 - 1$ and it is given by

$$D_\xi = d^2 + \frac{1}{2}\left[d, N^{+11}\right] + \hat{\sigma}^{11} + \hat{\omega}^{11} - \frac{1}{2}\left\{\alpha^{11}, \hat{\sigma}^{11}\right\} - \frac{1}{2}[d,[d,\alpha^{11}]] + O\left(G_F^2\right). \tag{3.29}$$

The second term gives the action in the bosonic ghost non singlet sector and the last two terms are mixing terms. In the degenerate mass case that we are considering here $\mathcal{M} = m_q \mathbf{1}$, the mixing terms are zero, since $\hat{\sigma}^{22}_{0a} = \hat{\sigma}^{22}_{a0} = 0$, so that the physical and non physical bosonic sectors decouple as it happens in the purely strong interaction case. Also, since $\hat{\sigma}^{22}_{ab} = M^2 \delta_{ab}$, the $\tilde{\xi}$ action reduces to the free action which gives a trivial multiplicative constant in the generating functional.

After these reductions the bosonic determinant in the non singlet sector reduces to the determinant of D_ξ. The ultraviolet divergent part of the integral over the ξ fields can be derived in closed form by regularizing the determinant in d dimensions and using standard heat–kernel techniques. The result reads as follows:

$$\frac{i}{2}\ln\det D_\xi = \frac{i}{2}\ln\det D_\xi|_{\text{strong}} + \frac{i}{2}\ln\det D_\xi|_{(8)} + \frac{i}{2}\ln\det D_\xi|_{(27)} + \frac{i}{2}\ln\det D_\xi|_{\Delta S=2}\,. \tag{3.30}$$

The first term was derived in ref. [10] with the same notation adopted here and we do not write it again. The remaining terms are the $\Delta S = \pm 1$ weak contributions in the octet and 27–plet sector and the $\Delta S = 2$ contribution, all in the presence of a singlet component of the meson field. We list in Appendix A the definition of the ultraviolet divergent octet W_i, $\bar{W}_i$ and 27-plet D_i, $\bar{D}_i$ weak operators, where $\bar{W}_i$ and $\bar{D}_i$ are new operators induced by the presence of the dynamical singlet field. The notation follows the one of ref. [5] for the octet case, with the exception of the $\bar{W}_i$ not introduced there and the eventual enlargement of the basis for $N > 3$. The calculation is done in $SU(N)$ with N generic, so that we maintain everywhere the explicit dependence upon the number of flavours. We find:

$$\begin{aligned}
\frac{i}{2}\ln\det D_\xi|_{(8)} = &-\frac{1}{(4\pi)^2(d-4)}\int dx\, \frac{N}{16}\Big\{ g_8'\left(W_5 - W_9 + W_{10} + W_{12} + W_{36}\right) + \\
&+ g_8\Big(\frac{8}{3}W_1 - \frac{2}{3}W_2 + W_5 + W_9 - W_{12} + \frac{1}{3}W_{14} + \frac{2}{3}W_{15} - \frac{1}{3}W_{16} - \frac{1}{6}W_{18} - \frac{5}{3}W_{19} + \\
&+ W_{20} + 2W_{21} + 2W_{22} + \frac{2}{3}W_{25} - W_{26} + \frac{1}{6}W_{27} - W_{36} - \frac{1}{6}W_{37} - W_{38}\Big) + \\
&+ \frac{1}{3}\bar{g}_8\left(2\bar{W}_1 + 2\bar{W}_2 + 3\bar{W}_3 - 2\bar{W}_5 - 6\bar{W}_8 - 12\bar{W}_9 + 2\bar{W}_{11} + 4\bar{W}_{15}\right)\Big\} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}g_8'\left(W_7+W_{11}-\overline{W}_3+\overline{W}_4\right)+g_8\left(\frac{1}{4}W_4+\frac{1}{8}W_6-\frac{3}{16}W_7+\frac{1}{16}W_8-\frac{1}{8}W_{11}-\right.\\
&-\frac{5}{24}\bar{W}_1+\frac{1}{6}\bar{W}_2+\frac{1}{16}\bar{W}_3-\frac{1}{8}\bar{W}_4-\frac{1}{24}\bar{W}_5+\frac{7}{24}\bar{W}_{11}+\frac{1}{12}\bar{W}_{15}+\frac{1}{8}\bar{W}_{17}-\frac{1}{8}\bar{W}_{13}-\\
&\left.-\frac{1}{48}\bar{W}_8-\frac{1}{24}\bar{W}_9-\frac{1}{8}\bar{W}_{12}+\frac{1}{8}\bar{W}_{16}-\frac{1}{12}\bar{W}_{14}-\frac{1}{24}\bar{W}_7+\frac{1}{24}\bar{W}_{10}\right)+\\
&+\bar{g}_8\frac{1}{8}\left(-\frac{1}{2}W_5-W_{10}+W_{38}+\bar{W}_{18}+2\bar{W}_{19}-\bar{W}_{20}-\bar{W}_{21}-\bar{W}_{22}+\bar{W}_{23}\right)+\\
&+\frac{1}{8N}\Big\{g_8\left(-W_5+2W_{10}+2W_{12}-4W_{21}-4W_{22}+2W_{36}+2W_{38}\right)-\\
&-2g_8'\left(W_{10}+W_{12}+W_{36}\right)+\bar{g}_8W_{11}\Big\}-\frac{1}{4N^2}(g_8-g_8')W_{11}+O(G_F^2)\,. \qquad (3.31)
\end{aligned}
$$

Notice that the octet determinant in the physical sector is the one in the presence of the singlet field. The inclusion of the singlet field modifies the standard ChPT determinant in three ways: the presence of the barred operators $\overline{W}_i$, the presence of new contributions to the W_i proportional to the singlet coupling $\bar{g}_8$ and an additional contribution to the W_i counterterms coming from the anticommutator term $\{\alpha^{11},\hat{\sigma}^{11}\}$ in eq. (3.29) when internal indices run through zero. The conversion from one realization of ChPT to the other can be done using the following formula

$$
\begin{aligned}
&\frac{i}{2}\ln\det D_\xi|_{(8)}=\frac{i}{2}\ln\det D_\xi|_{(8)}^{no\,singlet}-\frac{1}{(4\pi)^2(d-4)}\int dx\,\left\{g_8\left[\frac{1}{4N}W_{38}-\frac{1}{8N}W_5\,-\right.\right.\\
&\left.\left.-\frac{1}{4N}W_{10}+\frac{1}{4N^2}W_{11}\right]+(g_8,g_8',\bar{g}_8)\cdot\overline{W}_i\,\text{terms}+\overline{g}_8\cdot W_i\,\text{terms}\right\}+O(G_F^2)\,. \qquad (3.32)
\end{aligned}
$$

In actual computations of weak matrix elements with the singlet meson field integrated out, the appropriate divergence of the octet counterterms W_i, $i=5,10,11,38$ has to be taken. This divergence is the one listed in table 1 in the CHPT column. In the 27–plet sector we get

$$
\begin{aligned}
&\frac{i}{2}\ln\det D_\xi|_{(27)}=-\frac{1}{(4\pi)^2(d-4)}\int dx\;g_{27}\left\{\frac{N}{12}\left(D_1+D_2-\frac{1}{2}D_4+\frac{3}{2}D_8-D_{13}+\right.\right.\\
&\left.+D_{14}-\frac{1}{2}D_{15}-D_{16}+D_{17}+D_{18}+2D_{21}+2D_{22}\right)+\frac{1}{12}D_1+\frac{5}{6}D_2-\frac{1}{8}D_3-\\
&-\frac{7}{24}D_4-\frac{1}{4}D_5+\frac{1}{4}D_6+\frac{1}{2}D_7+\frac{3}{8}D_8-\frac{3}{8}D_9+\frac{1}{4}D_{10}-\frac{1}{4}D_{11}-\frac{1}{8}D_{12}-\\
&-\frac{1}{12}D_{13}+\frac{1}{12}D_{14}-\frac{1}{24}D_{15}-\frac{1}{12}D_{16}+\frac{7}{12}D_{17}+\frac{7}{12}D_{18}-\frac{1}{4}D_{19}-\frac{1}{4}D_{20}+\frac{1}{6}D_{21}+\\
&\left.+\frac{1}{6}D_{22}+\frac{1}{4}D_{23}+\frac{1}{4}D_{24}-\frac{1}{4}\bar{D}_1-\frac{1}{4}\bar{D}_2-\frac{1}{4}\bar{D}_3+\frac{1}{4N}D_{12}\right\}+\mathcal{O}(G_F^2)\,, \qquad (3.33)
\end{aligned}
$$

where the appropriate tensor $t^{ij,kl}$ to $\Delta S=1$ or 2 interactions has to be taken inside the D_i and $\bar{D}_i$ counterterms defined in Appendix A. This time the presence of the singlet field modifies the standard ChPT determinant only via the presence of the $\bar{D}_i$ counterterms.

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3.3 Integral over the singlet fields

After the shift of the $\overline{X}$ field as in eq. (3.23) the operator acting on X_0 can be written as follows:

$$X_0^T \overline{D}_X X_0 = X_0^T \left[D_X - B^T \bar{D}^{-1} B \right] X_0 \,, \tag{3.34}$$

where

$$\begin{aligned}
D_X &= D_X^0 + A \,, \\
D_X^0 &= \tau_3 \left(\Box + M^2 \right) + \frac{N}{3}(1-\tau_1) \left(\alpha \Box + m_0^2 \right) \,, \\
A &= A_s + A_w \,, \\
A_s &= \frac{1}{4N}(1+\tau_3)\langle \hat{\chi}_+ \rangle - N(1-\tau_1) \left(v_1 \langle u_\mu u^\mu \rangle + v_2 \langle \hat{\chi}_+ \rangle \right) + O(\Phi_0^2) \,, \\
A_w &= \frac{1}{4N}(1+\tau_3)(g_8' - g_8) k \langle \Delta \chi_+ \rangle - \left(\frac{1+\tau_3}{2} - \frac{\tau_1}{2} \right) \frac{\bar{g}_8}{4} k \langle \Delta \chi_+ \rangle - \\
&\quad - N(1-\tau_1) \Big(\tilde{v}_8 k \langle \Delta u_\mu u^\mu \rangle + \tilde{v}_5 k \langle \Delta \chi_+ \rangle + \tilde{v}_{27} q \langle \Delta u_\mu \rangle \langle \Delta u^\mu \rangle + \\
&\quad + \tilde{v}_0 k \langle \Delta u_\mu \rangle \langle u^\mu \rangle \Big) + O(\Phi_0^2) \,, \\
B &= B_s + B_w \,, \\
B_s^a &= \frac{1+\tau_3}{2} \frac{1}{2\sqrt{2N}} \langle \lambda^a \chi_+ \rangle \,, \\
B_w^a &= \frac{1+\tau_3}{2} \frac{1}{4\sqrt{2N}} \Big[g_8 k \Big(-i \langle [d_s^\mu, [\Delta, u_\mu]] \lambda^a \rangle - 2 \left\langle d_s^2 \Delta \lambda^a \right\rangle - \langle \{ \Delta, \chi_+ \} \lambda^a \rangle + \\
&\qquad + \frac{1}{2} \langle [u^\mu, [\Delta, u_\mu]] \lambda^a \rangle \Big) + g_8' k \Big(\langle \{ \Delta, \chi_+ \} \lambda^a \rangle - \langle [\Delta, \chi_-] \lambda^a \rangle \Big) - \\
&\qquad - \frac{\bar{g}_8}{2} k \langle \Delta \lambda^a \rangle \langle \hat{\chi}_+ \rangle - g_{27} q \langle \Delta \lambda^a \rangle \langle \Delta \chi_+ \rangle \Big] + \left(\frac{1+\tau_3}{2} - \frac{\tau_1 + i\tau_2}{2} \right) \times \\
&\quad \times \frac{N}{4\sqrt{2N}} \Big[-2 g_8 k \langle \Delta \lambda^a \rangle \left(\frac{1}{3} \left(\alpha \Box + m_0^2 \right) - v_1 \langle u^2 \rangle - v_2 \langle \hat{\chi}_+ \rangle \right) + \\
&\qquad + \bar{g}_8 k \Big(\frac{i}{2} \left\langle \{ d_s^\mu, [\Delta, u_\mu] \} \lambda^a \right\rangle - \frac{i}{2} \left\langle [d_s^\mu, [\Delta, u_\mu]] \lambda^a \right\rangle - \langle d_s^2 \Delta \lambda^a \rangle - \\
&\qquad - M^2 \langle \Delta \lambda^a \rangle + \frac{1}{4} \left\langle [u^\mu, [\Delta, u_\mu]] \lambda^a \right\rangle - \frac{1}{4} \left\langle \{ \Delta, \chi_+ \} \lambda^a \right\rangle \Big) \Big] \,.
\end{aligned} \tag{3.35}$$

The covariant derivative d_s^μ is the one defined in eq. (3.5). We separated the singlet operator A and the mixing operator B in their strong interacting part (with subscript s) and their weak interacting part up to order G_F. We recall that $\langle \hat{\chi}_+ \rangle = \langle \chi_+ \rangle - 2NM^2$ in the degenerate mass case. As also in the analysis of the purely strong interacting sector we cannot apply straightforwardly the heat–kernel techniques, because the differential operator does not reduce to a diagonal Klein–Gordon operator when the external fields are put to zero. Therefore we just expand the logarithm of the differential operator, and calculate only the ultraviolet divergent terms. The expansion gives:

$$\frac{i}{2}\mathrm{Tr}\ln\left(\overline{D}_X/D_X^0\right) = \frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}\left(\overline{D}_X - D_X^0\right)\right] - \\ -\frac{i}{4}\mathrm{Tr}\left[D_X^{0\ -1}\left(\overline{D}_X - D_X^0\right)D_X^{0\ -1}\left(\overline{D}_X - D_X^0\right)\right] + \cdots\,, \quad (3.36)$$

where the ellipsis contains ultraviolet finite terms only. The inverse of the "free" operator D_X^0 is:

$$D_X^{0\ -1} = G_0\left[\tau_3 - (1+\tau_1)\frac{N}{3}\left(\alpha\Box + m_0^2\right)G_0\right]\,, \quad (3.37)$$

where

$$\left(\Box + M^2\right)_x G_0(x-y) = \delta(x-y)\,, \quad (3.38)$$

and

$$\overline{D}_X - D_X^0 = A - B^T\bar{D}^{-1}B\,. \quad (3.39)$$

While the strong interacting part has been fully derived in ref. [10], the ultraviolet divergences of the singlet determinant at order G_F in the expansion in powers of the weak coupling, are given by the following terms:

$$\frac{i}{2}\mathrm{Tr}\ln\left(\overline{D}_X/D_X^0\right) = \frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}A_w\right] - \frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}A_s D_X^{0\ -1}A_w\right] \\ -\frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}B_s^T\bar{D}^{-1}B_w + D_X^{0\ -1}B_w^T\bar{D}^{-1}B_s\right] \\ +\frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}A_s D_X^{0\ -1}\left(B_s^T\overline{D}^{-1}B_w + B_w^T\overline{D}^{-1}B_s\right)\right] + O(G_F^2)\,. \quad (3.40)$$

The inverse of the non singlet operator $\overline{D}$ is expanded around its free part. In the third term of eq. (3.40) we need its expansion up to order G_0^2 and keeping only its strongly interacting part (since we stop at order G_F). This gives:

$$\overline{D}_{ab}^{-1} = G_0\tau_3\delta_{ab} + G_0\left(M^2\delta_{ab}\tau_3 - \hat{\sigma}_{ab}^{11}\frac{1+\tau_3}{2} - \hat{\sigma}_{ab}^{22}\frac{1-\tau_3}{2}\right)G_0 + O(G_0^3)\,. \quad (3.41)$$

Note that the $O(G_0^2)$ term of $\overline{D}_{ab}^{-1}$ in the third term of eq. (3.40) yields ultraviolet divergences only when the term proportional to $\alpha\Box$ in B_w is considered. The same is true for the last term in eq. (3.40), where $\overline{D}^{-1}$ is only taken up to order G_0. For the sake of clearness we list in Appendix B the ultraviolet divergent contributions produced by each term of eq. (3.40). The total ultraviolet divergent contribution given by the integral over the singlet fields at order G_F in the weak interaction sector is given by:

$$\frac{i}{2}\mathrm{Tr}\ln\left(\overline{D}_X/D_X^0\right) = -\frac{1}{(4\pi)^2(d-4)}\int dx\,\left\{\frac{1}{6}m_0^2(g_8' - 2g_8)k\langle\Delta\chi_+\rangle + \right. \\ +\frac{\alpha^2}{36}(g_8' - g_8)k\langle\Delta\chi_+\rangle\langle\hat{\chi}_+\rangle + \frac{\alpha}{24}\left[-2g_8'\left(W_{10} + W_{12} + W_{36}\right) + \right.$$

$$
\begin{aligned}
&+g_8\Big(4W_{10}+2W_{12}-4W_{21}-4W_{22}+2W_{36}\Big)+2g_{27}D_{12}+\\
&+2\bar{g}_8 k\langle\Delta\chi_+\rangle\langle\hat{\chi}_+\rangle\Big]+\frac{1}{16}\bar{g}_8\Big(W_5-2W_{12}+4W_{21}+4W_{22}-\\
&-2W_{36}-2W_{38}\Big)-\frac{1}{2}\Big(\tilde{v}_8 k\langle\Delta u_\mu u^\mu\rangle+\tilde{v}_5 k\langle\Delta\chi_+\rangle+\\
&+\tilde{v}_{27} q\langle\Delta u_\mu\rangle\langle\Delta u^\mu\rangle+\tilde{v}_0 k\langle\Delta u_\mu\rangle\langle u^\mu\rangle\Big)\langle\hat{\chi}_+\rangle-\\
&-\frac{1}{2}(g_8'-2g_8)k\langle\Delta\chi_+\rangle\Big(v_1\langle u_\mu u^\mu\rangle+v_2\langle\hat{\chi}_+\rangle\Big)+\\
&+\frac{1}{8N}\Big[2g_8'(W_{10}+W_{12}+W_{36})+g_8\Big(W_5-2W_{10}-2W_{12}+\\
&+4W_{21}+4W_{22}-2W_{36}-2W_{38}\Big)-\bar{g}_8 W_{11}-2g_{27}D_{12}\Big]+\\
&+\frac{1}{4N^2}(g_8-g_8')W_{11}\Big\}+O(G_F^2)\,. \qquad (3.42)
\end{aligned}
$$

The most relevant behaviour concerns the appearance of the quenched chiral logarithms, i.e. of the type $m_0^2\log m_\pi^2$, that are pure artefacts of the quenched approximation. As it was discussed at length in ref. [10] quenched chiral logarithms appearing in the strong sector can be formally reabsorbed in a redefinition (in fact a true renormalization in dimensional regularization) of the B_0 parameter. In the weak sector an analogous mechanism occurs. The quenched chiral logarithms which appear through the first term in eq. (3.42) can be formally reabsorbed into a redefinition of the weak mass term coupling g_8' of the leading order Lagrangian (2.4). To remove the m_0^2 divergence in eq. (3.42) one has to add to the lowest order parameter g_8' in the leading Lagrangian (2.4) a d-dependent part proportional to m_0^2 that has a pole at d=4:

$$
g_8'\to g_8'\left[1+\frac{\mu^{d-4}}{16\pi^2}\frac{1}{d-4}\frac{2m_0^2}{3F^2}\left(1-2\frac{g_8}{g_8'}\right)+\delta g_8'(\mu)\right], \qquad (3.43)
$$

so that the renormalized coupling can be defined as follows:

$$
g_{8R}'=g_8'\left(1-\frac{m_0^2}{48\pi^2F^2}\left(1-2\frac{g_8}{g_8'}\right)\log\frac{M^2}{\mu^2}+\delta g_8'(\mu)\right). \qquad (3.44)
$$

The rescaling of the coupling g_8' together with the rescaling of the parameter $B_0\to\bar{B}_0$ defined in ref. [10] in the tree level contribution to any weak observable can be used as a short-cut procedure to unreveal the presence of quenched chiral logarithms, generated when the quenched approximation is implemented to one loop.

3.4 The bosonic determinant: complete result

The complete contribution to the bosonic part of the logarithm of the quenched generating functional to one loop is given by

$$
Z^b_{1\,\mathrm{loop}}=\frac{i}{2}\ln\det\bar{D}+\frac{i}{2}\mathrm{Tr}\ln\bar{D}_X+\frac{i}{2}\mathrm{Tr}\ln(G\tau_3)\,, \qquad (3.45)
$$

where the first term is the non singlet contribution which reduces to eqs. (3.31) and (3.33) in the degenerate mass case, the second term is the singlet part given by eq. (3.42) and the last term comes from the Jacobian of the transformation induced by the weak connection. It gives:

$$\begin{aligned} \frac{i}{2}\mathrm{Tr}\ln(G\tau_3) &= \frac{i}{2}\mathrm{Tr}\left(\alpha_{ab}\tau_3\right)+O(G_F^2) \\ &= \frac{i}{2}\int dx\,\alpha^{11}_{aa}(x)+O(G_F^2) \\ &= \frac{i}{2}\int dx\,\Big(Nk\langle\Delta\rangle+q\langle\Delta^2\rangle+\bar{g}_8k\langle\Delta\rangle\Big)+O(G_F^2)\,, \end{aligned} \tag{3.46}$$

which is zero at order G_F since the projection operator Δ_{ij} is traceless.

It is easy to verify that the complete cancellation of the $1/N, 1/N^2$ terms is provided by the sum of the non-singlet contributions (3.31), (3.33) and the singlet contributions (3.42), a feature that was already pointed out in ref. [10] for the strong sector. This cancellation is a consequence of the introduction of a dynamical singlet field, independently of whether the quenched approximation is made. Within the bosonic sector, quenching effects are of two types: 1) terms proportional to m_0^2 and α and 2) the substitution $\langle\chi_+\rangle\to\langle\hat{\chi}_+\rangle$. Terms of type 1) originate from the double-pole part of the super-η' two-point function, whereas the substitution of type 2) does eliminate the linear flavour number dependence that comes from $\langle\mathcal{M}\rangle$ in the degenerate mass case. By eliminating terms of type 1) and replacing back $\langle\chi_+\rangle$ the bosonic determinant $Z^b_{1\,\mathrm{loop}}$ becomes the full ChPT determinant with a dynamical singlet field.

3.5 The fermionic ghost sector

The differential operator D_ζ as defined in eq. (3.2) is given by $(a,b=0,1,\ldots N^2-1)$

$$D_{\zeta ab}=d_\mu\bar{G}_{ab}d^\mu-\frac{1}{2}\left\{\bar{N}^-_{ab},d\right\}+\frac{1}{2}\left[d,\bar{N}^+_{ab}\right]+\bar{\sigma}_{ab}+\bar{\omega}_{ab}\,, \tag{3.47}$$

where $\bar{G}_{ab}=\delta_{ab}+\bar{\alpha}_{ab}$, d_μ is the covariant derivative acting on the ζ field as defined in the purely strong interaction case and the barred quantities are defined as follows

$$\begin{aligned} \bar{\alpha}_{ab} &= \frac{g_8}{2}k\left\langle\Delta\hat{\lambda}_a\hat{\lambda}_b\right\rangle\,, \\ \bar{N}^-_{\mu ab} &= g_8\frac{i}{4}k\left\langle\{\Delta,u_\mu\}\hat{\lambda}_a\hat{\lambda}_b\right\rangle+g_{27}iq\left\langle\Delta\hat{\lambda}_a\hat{\lambda}_b\right\rangle\langle\Delta u_\mu\rangle\,, \\ \bar{N}^+_{\mu ab} &= -\frac{i}{4}g_8k\left\langle[\Delta,u_\mu]\hat{\lambda}_a\hat{\lambda}_b\right\rangle\,, \\ \bar{\omega}_{ab} &= \frac{g_8}{8}k\left\langle(\{\Delta,u^2\}+u_\mu\Delta u^\mu)\hat{\lambda}_a\hat{\lambda}_b\right\rangle+\frac{g_{27}}{4}q\left\langle\{\Delta,u_\mu\}\hat{\lambda}_a\hat{\lambda}_b\right\rangle\langle\Delta u^\mu\rangle+ \\ &\quad+\frac{g_8'}{8}k\left(\left\langle\{\Delta,\chi_+\}\hat{\lambda}_a\hat{\lambda}_b\right\rangle-\left\langle[\Delta,\chi_-]\hat{\lambda}_a\hat{\lambda}_b\right\rangle\right)\,, \\ \bar{\sigma}_{ab} &= \frac{1}{4}\left\langle\left(u_\mu u^\mu+\chi_++4B_0\mathcal{M}\right)\hat{\lambda}^a\hat{\lambda}^b\right\rangle\,. \end{aligned} \tag{3.48}$$

After rescaling the coefficient of the double derivative term to unit, keeping up to order G_F terms and redefining the covariant derivative in the presence of weak interactions (i.e. by reabsorbing the $\bar{N}^-_\mu$ term in the covariant derivative d_μ), the fermionic ghost determinant is now given by the product $\det D_\zeta = \det \bar{G} \cdot \det D'_\zeta$, where the rescaled operator D'_ζ acts on the fermionic ghost fields as follows

$$
\begin{aligned}
D'_{\zeta ab}\zeta^b &= d_\mu d^\mu \zeta_a + \left(\frac{1}{2}\left[d, \bar{N}^+_{ab}\right] + \bar{\sigma}_{ab} + \bar{\omega}_{ab} - \frac{1}{2}\{\bar{\alpha}, \bar{\sigma}\}_{ab} - \frac{1}{2}[d, [d, \bar{\alpha}_{ab}]]\right)\zeta^b \\
d^\mu \zeta_a &= \partial^\mu \zeta_a + \bar{\Gamma}^\mu_{ab}\zeta^b\,,
\end{aligned}
\tag{3.49}
$$

with

$$
\bar{\Gamma}^\mu_{ab} = -\langle \Gamma^\mu \hat{\lambda}_a \hat{\lambda}_b \rangle - \frac{1}{2}\bar{N}^{-\mu}_{ab}\,. \tag{3.50}
$$

The result for the ultraviolet divergences of the fermionic ghost determinant at one loop reads as follows:

$$
\begin{aligned}
i \ln \det D_\zeta &= i \ln \det D'_\zeta|_{\text{strong}} + i \ln \det D'_\zeta|_{(8)} + i \ln \det D'_\zeta|_{(27)} + \\
&\quad + i \ln \det D'_\zeta|_{\Delta S=2} + i \mathrm{Tr} \ln \bar{G}\,,
\end{aligned}
\tag{3.51}
$$

where the first contribution from strong interactions has been given in ref. [10], while the remaining contributions from the weak sector at order G_F are

$$
\begin{aligned}
i \ln \det D'_\zeta|_{(8)} = -\frac{1}{(4\pi)^2(d-4)} \int dx \Big\{ D^8_\xi(N) + \frac{NM^2}{8}\Big[g_8 k \langle \Delta u^2 \rangle + 2\,(g'_8 - g_8)\, k \langle \Delta \chi_+ \rangle \\
+ 2 \bar{g}_8 k \langle \Delta u_\mu \rangle \langle u^\mu \rangle \Big]\Big\} + O(G_F^2)
\end{aligned}
\tag{3.52}
$$

and for the 27-plet

$$
i \ln \det D'_\zeta|_{(27)} = -\frac{1}{(4\pi)^2(d-4)} \int dx \Big\{ D^{27}_\xi(N) + \frac{NM^2}{2} g_{27} q \langle \Delta u_\mu \rangle \langle \Delta u^\mu \rangle \Big\} + O(G_F^2)\,. \tag{3.53}
$$

With the notation $D^8_\xi(N)$ and $D^{27}_\xi(N)$ we define the part of the integrals over the physical ξ field in eqs. (3.31) and (3.33) that carries the linear flavour number dependence. The term $i \ln \det D'_\zeta|_{\Delta S=2}$ is the same as in eq. (3.53) where the appropriate tensor $t^{ij,kl}$ is used. The contribution from the Jacobian term $i\mathrm{Tr} \ln \bar{G}$ is again zero at order G_F, since it is given by

$$
\begin{aligned}
i\mathrm{Tr} \ln \bar{G} &= i \mathrm{Tr}\, \bar{\alpha}_{ab} + O(G_F^2) \\
&= i \int dx\ \bar{\alpha}_{aa} + O(G_F^2) \\
&= i \int dx\ \frac{g_8}{2} N k \langle \Delta \rangle + O(G_F^2)
\end{aligned}
\tag{3.54}
$$

and Δ_{ij} is traceless. The way eqs. (3.52) and (3.53) have been written shows explicitly that the integral over the fermionic ghost fields generates a linear flavour number dependence only, as expected. As it was noted in ref. [10], the linear flavour number dependence of the physical determinant is not fully explicit for degenerate quark masses. In fact, the terms in addition to $D^8_\xi(N)$ and $D^{27}_\xi(N)$ in eqs. (3.52), (3.53) guarantee the cancellation of those N dependent contributions generated in the physical sector by the $\langle\chi_+\rangle$ in the degenerate mass case.

4. Complete result

In the analysis of the complete result for the ultraviolet divergences of the weak generating functional to one loop, with and without the quenched approximation, we limit to the case of non-singlet Green's functions. This corresponds to neglecting the barred chiral invariants $\overline{W}_i$ and $\overline{D}_i$ which we previously included in the partial contributions. They should be considered whenever singlet Green's functions are involved. We also remind the reader that the calculation of singlet Green's functions should also involve a set of contributions proportional to $\overline{\Phi}_0$ ($\overline{\phi}_0$ in the unquenched case) and coming from the expansion of the strong V_i and weak $\tilde{V}_i$ potentials. For the sake of clearness we limit to the non-singlet Green's functions that are the most relevant to phenomenological applications in the weak sector.

In the following formulas we give the complete ultraviolet divergent part of the hadronic weak generating functional (with the exclusion of singlet operators) in the quenched approximation and with degenerate quark masses for $\Delta S = \pm 1$ interactions, octet and 27-plet, and for $\Delta S = 2$ interactions. They are as follows:

$$
\begin{aligned}
Z^q_{(8)} = & -\frac{1}{(4\pi)^2(d-4)}\int dx\,\left\{\frac{1}{6}m_0^2(g'_8-2g_8)k\langle\Delta\chi_+\rangle+\frac{1}{4}g_8W_4+\right.\\
&+\frac{1}{8}W_6+\left[-\frac{3}{16}g_8\left(1-\frac{16}{3}v_1\right)+\frac{1}{8}g'_8\left(1-4v_1\right)\right]W_7+\frac{1}{16}(g_8-8\tilde{v}_8)W_8+\\
&+\left(\frac{\alpha}{12}(g_8-g'_8)-\frac{1}{8}\bar{g}_8\right)(W_{12}+W_{36})+\left(\frac{\alpha}{12}(2g_8-g'_8)-\frac{1}{8}\bar{g}_8\right)W_{10}+\\
&+\left[\left(\frac{1}{8}+\frac{\alpha^2}{36}\right)(g'_8-g_8)-\frac{1}{2}v_2(g'_8-2g_8)-\frac{1}{2}\tilde{v}_5+\frac{\alpha}{12}\bar{g}_8\right]W_{11}+\\
&\left.+\left(-\frac{\alpha}{6}g_8+\frac{1}{4}\bar{g}_8\right)(W_{21}+W_{22})\right\}+O(G_F^2)\,,
\end{aligned}
\tag{4.1}
$$

where counterterms W_i are listed in Appendix A and the operators W_8 and W_{11} have been replaced by the quenched ones with the substitution $\langle\chi_+\rangle\to\langle\hat{\chi}_+\rangle$. As explained in the case of strong interactions [10], the redefinition provides the subtraction of terms that become linearly dependent upon the number of flavours in the degenerate quark mass limit. The 27-plet contribution is given by

$$
Z^q_{(27)} = -\frac{1}{(4\pi)^2(d-4)}\int dx\,g_{27}\left\{\frac{1}{12}D_1+\frac{5}{6}D_2-\frac{1}{8}D_3-\frac{7}{24}D_4-\frac{1}{4}D_5+\right.
$$

$$
\begin{aligned}
&+\frac{1}{4}D_6+\frac{1}{2}D_7+\frac{3}{8}D_8-\frac{3}{8}D_9+\frac{1}{4}\left(1-2\frac{\tilde{v}_{27}}{g_{27}}\right)D_{10}-\frac{1}{4}D_{11}-\\
&-\frac{1}{8}\left(1-\frac{2}{3}\alpha\right)D_{12}-\frac{1}{12}D_{13}+\frac{1}{12}D_{14}-\frac{1}{24}D_{15}-\frac{1}{12}D_{16}+\frac{7}{12}D_{17}+\\
&+\frac{7}{12}D_{18}-\frac{1}{4}D_{19}-\frac{1}{4}D_{20}+\frac{1}{6}D_{21}+\frac{1}{6}D_{22}+\\
&+\frac{1}{4}D_{23}+\frac{1}{4}D_{24}\Big\}+O(G_F^2)\,, \qquad (4.2)
\end{aligned}
$$

where counterterms D_i are listed in Appendix A and the operator D_{10} has been replaced by the quenched one with $\langle\chi_+\rangle \to \langle\hat{\chi}_+\rangle$. The quenched generating functional for $\Delta S = 2$ interactions has exactly the same structure of $Z^q_{(27)}$, where the $\Delta S = 1$ tensor $t^{ij,kl}$ in the D_i counterterms is replaced by the $\Delta S = 2$ one.

4.1 Analysis of divergent counterterms

In order to make clear within the present approach how the quenched approximation modifies phenomenological predictions in the weak sector we first investigate the main properties of the weak divergent counterterms in the full theory and their rôle in mediating weak processes. We then compare the quenched value of each divergent counterterm to the corresponding one in the full theory. Notice that once the quenched counterpart of each divergent contribution has been derived, one knows how the quenched approximation affects the coefficient of chiral logarithms for any weak observable.

In the full theory we define the divergent contribution to the $\Delta S = \pm 1$ counterterm Lagrangian at order p^4 with the following sums

$$
\mathcal{L}^4_{(8)} = g_8 \sum_{\{i\}} w_i W_i\,, \qquad (4.3)
$$

$$
\mathcal{L}^4_{(27)} = g_{27} \sum_{i=1}^{24} d_i D_i\,, \qquad (4.4)
$$

where the set of values $\{i\}$ in the octet Lagrangian runs over the divergent set of counterterms given in Appendix A. They are 25 in total. The 27-plet tensor $t^{ij,kl}$ inside the D_i invariants is the one defined in eq. (2.8). The counterterm Lagrangian in the $\Delta S = 2$ case is

$$
\mathcal{L}^4_{\Delta S=2} = g_{27}^{\Delta S=2} \sum_{i=1}^{24} d_i D_i\,, \qquad (4.5)
$$

where now $t^{23,23} = t^{32,32} = 1$ and 0 otherwise. Since the 27-plet operators which induce $\Delta S = 1$ and $\Delta S = 2$ transitions are components of the same irreducible tensor under $SU(3)_L \times SU(3)_R$, the coefficients d_i in both Lagrangians have to be the same. The renormalized value of the coefficients w_i and d_i can be defined in the conventional way and using dimensional regularization:

$$\begin{aligned}
w_i &= \left(\upsilon_i + \frac{g_8'}{g_8}\upsilon_i' + \frac{\bar{g}_8}{g_8}\bar{\upsilon}_i\right)\lambda + w_i^r(\mu) \equiv \nu_i\,\lambda + w_i^r(\mu)\,, \\
d_i &= \delta_i\,\lambda + d_i^r(\mu)\,,
\end{aligned} \tag{4.6}$$

while we recall the analogous definition for the coefficients of the strong counterterms [13]

$$L_i = \Gamma_i\,\lambda + L_i^r(\mu)\,. \tag{4.7}$$

λ contains the divergence at $d = 4$

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left[\frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1\right)\right], \tag{4.8}$$

μ is the renormalization scale, while the coefficients υ_i, υ_i', $\bar{\upsilon}_i$, δ_i and their quenched counterpart υ_i^q, $\upsilon_i'^q$, $\bar{\upsilon}_i^q$, δ_i^q are given in tables 1 and 2. The coefficients Γ_i of the strong counterterms and their quenched counterpart can be found in ref. [10]. For the following phenomenological analysis it is also useful to introduce the scale independent constants

$$\begin{aligned}
\bar{w}_i &= \frac{32\pi^2}{\nu_i} w_i^r(\mu) - \ln\frac{M^2}{\mu^2}\,, \\
\bar{d}_i &= \frac{32\pi^2}{\delta_i} d_i^r(\mu) - \ln\frac{M^2}{\mu^2}\,, \\
\bar{L}_i &= \frac{32\pi^2}{\Gamma_i} L_i^r(\mu) - \ln\frac{M^2}{\mu^2}
\end{aligned} \tag{4.9}$$

and their quenched counterparts $\bar{w}_i^q$, $\bar{d}_i^q$, $\bar{L}_i^q$. They carry the chiral logarithms, e.g. $\bar{w}_i = -\ln M^2 + \ldots$, where M^2 is the squared bare meson mass, and the analogous definition for the other constants.

The quenched approximation largely reduces the ultraviolet divergent contribution to the octet sector at one loop. The only octet operators whose divergence is not modified by quenching are W_i, $i = 4, 6$. Within the class of divergences proportional to g_8, the octet operators W_i, $i = 10, 12, 21, 22, 36$ acquire a divergence proportional to α coming from the quenched anomalous singlet sector and W_{11} gets a contribution proportional to α^2. W_7, W_8 and W_{11} get a contribution from the strong and weak potentials υ_i, $\tilde{\upsilon}_i$. Notice that our list of octet counterterms in the full theory differs from the one in ref. [5] for the presence of W_{38}, which has to be kept as a linearly independent operator for a flavour group $SU(N)$ with generic N. For $N = 3$ flavours this operator can be eliminated with the use of Cayley-Hamilton relations, that give $W_{38} = -W_5 + W_6 + 1/2W_7 + W_8$.

The octet operators W_i contribute to several different decay processes of the K meson. $W_1, \ldots W_4$ contribute to the $K \to 3\pi$ decays, $W_5, \ldots W_{12}$ and W_{38} to

W_i	v_i CHPT	v_i qCHPT $\langle\chi_+\rangle \to \langle\hat\chi_+\rangle$	v'_i CHPT	v'_i qCHPT $\langle\chi_+\rangle \to \langle\hat\chi_+\rangle$	$\bar v_i$ qCHPT $\langle\chi_+\rangle \to \langle\hat\chi_+\rangle$
1	$\frac{N}{6}$	0	0	0	0
2	$-\frac{N}{24}$	0	0	0	0
4	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0
5	$\frac{N}{16}$	0	$\frac{N}{16}$	0	0
6	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0
7	$-\frac{3}{16}$	$-\frac{3}{16}\left(1-\frac{16}{3}v_1\right)$	$\frac{1}{8}$	$\frac{1}{8}(1-4v_1)$	0
8	$\frac{1}{16}$	$\frac{1}{16}\left(1-8\frac{\tilde v_8}{g_8}\right)$	0	0	0
9	$\frac{N}{16}$	0	$-\frac{N}{16}$	0	0
10	$\frac{1}{2N}$	$\frac{\alpha}{6}$	$\frac{N}{16}-\frac{1}{4N}$	$-\frac{\alpha}{12}$	$-\frac{1}{8}$
11	$-\frac{1}{8}-\frac{1}{2N^2}$	$-\frac{1}{8}-\frac{\alpha^2}{36}+v_2$	$\frac{1}{8}+\frac{1}{4N^2}$	$\frac{1}{8}+\frac{\alpha^2}{36}-\frac{v_2}{2}-\frac{\tilde v_5}{2g'_8}$	$\frac{\alpha}{12}$
12	$-\frac{N}{16}+\frac{1}{4N}$	$\frac{\alpha}{12}$	$\frac{N}{16}-\frac{1}{4N}$	$-\frac{\alpha}{12}$	$-\frac{1}{8}$
14	$\frac{N}{48}$	0	0	0	0
15	$\frac{N}{24}$	0	0	0	0
16	$-\frac{N}{48}$	0	0	0	0
18	$-\frac{N}{96}$	0	0	0	0
19	$-\frac{5}{48}N$	0	0	0	0
20	$\frac{N}{16}$	0	0	0	0
21	$\frac{N}{8}-\frac{1}{2N}$	$-\frac{\alpha}{6}$	0	0	$\frac{1}{4}$
22	$\frac{N}{8}-\frac{1}{2N}$	$-\frac{\alpha}{6}$	0	0	$\frac{1}{4}$
25	$\frac{N}{24}$	0	0	0	0
26	$-\frac{N}{16}$	0	0	0	0
27	$\frac{N}{96}$	0	0	0	0
36	$-\frac{N}{16}+\frac{1}{4N}$	$\frac{\alpha}{12}$	$\frac{N}{16}-\frac{1}{4N}$	$-\frac{\alpha}{12}$	$-\frac{1}{8}$
37	$-\frac{N}{96}$	0	0	0	0
38	$-\frac{N}{16}$	0	0	0	0

Table 1: List of the octet counterterms W_i at order p^4 for a generic number of flavours N. For each counterterm we list the coefficients of the divergent part v_i, v'_i and $\bar v_i$ as defined in eq. (4.6) in the unquenched case with no singlet field (denoted as CHPT) and in the quenched case (qCHPT). The unquenched contribution from $\bar v_i$ is always zero in standard CHPT where no dynamical singlet field is present. As we have indicated in the table, chiral invariants containing $\langle\chi_+\rangle$ have to be changed with $\langle\chi_+\rangle \to \langle\hat\chi_+\rangle$ in the quenched case.

$K \to 2\pi$, 3π decays, $W_{14}, \dots W_{18}$ to radiative K decays and finally $W_{19}, \dots W_{27}$ only to processes which involve an external W gauge boson (in addition to the non leptonic weak transition). W_{36} and W_{37} are contact terms which are only needed for renormalization purposes.

The rôle of the contributions induced by the weak mass term (i.e. the terms proportional to g_8') has been extensively discussed in ref. [15] for the most general case of off-shell Green's functions, while in ref. [5] it is shown how their contribution can be reabsorbed in a rescaling of some of the counterterms proportional to g_8 for on-shell matrix elements. The operators W_i, $i = 5, 7, 9, 10, 11, 12, 36$ get a contribution proportional to g_8' in the full theory. In the quenched approximation the contribution to W_5 and W_9 becomes zero, while W_{10}, W_{12}, W_{36} get a contribution proportional to α, W_{11} to α^2 and W_7, W_{11} to v_i, $\tilde{v}_i$.

We also included the contributions coming from the extra term proportional to $\bar{g}_8$ of the leading $O(p^2)$ weak Lagrangian in the presence of the singlet field. This term induces contributions to the octet operators W_i, $i = 10, 11, 12, 21, 22, 36$, in addition to all the singlet induced barred operators $\overline{W}_i$. After quenching, only the divergence of W_{11} is modified (with the exclusion of the $\bar{W}_i$ operators), where a contribution proportional to α appears.

D_i	δ_i CHPT	δ_i qCHPT $\langle\chi_+\rangle \to \langle\hat{\chi}_+\rangle$
1	$\frac{N}{12} + \frac{1}{12}$	$\frac{1}{12}$
2	$\frac{N}{12} + \frac{5}{6}$	$\frac{5}{6}$
3	$-\frac{1}{8}$	$-\frac{1}{8}$
4	$-\frac{N}{24} - \frac{7}{24}$	$-\frac{7}{24}$
5	$-\frac{1}{4}$	$-\frac{1}{4}$
6	$\frac{1}{4}$	$\frac{1}{4}$
7	$\frac{1}{2}$	$\frac{1}{2}$
8	$\frac{N}{8} + \frac{3}{8}$	$\frac{3}{8}$
9	$-\frac{3}{8}$	$-\frac{3}{8}$
10	$\frac{1}{4}$	$\frac{1}{4}\left(1 - 2\frac{\tilde{v}_{27}}{g_{27}}\right)$
11	$-\frac{1}{4}$	$-\frac{1}{4}$
12	$-\frac{1}{8} + \frac{1}{4N}$	$-\frac{1}{8}\left(1 - \frac{2}{3}\alpha\right)$
13	$-\frac{N}{12} - \frac{1}{12}$	$-\frac{1}{12}$
14	$\frac{N}{12} + \frac{1}{12}$	$\frac{1}{12}$
15	$-\frac{N}{24} - \frac{1}{24}$	$-\frac{1}{24}$
16	$-\frac{N}{12} - \frac{1}{12}$	$-\frac{1}{12}$
17	$\frac{N}{12} + \frac{7}{12}$	$\frac{7}{12}$
18	$\frac{N}{12} + \frac{7}{12}$	$\frac{7}{12}$
19	$-\frac{1}{4}$	$-\frac{1}{4}$
20	$-\frac{1}{4}$	$-\frac{1}{4}$
21	$\frac{N}{6} + \frac{1}{6}$	$\frac{1}{6}$
22	$\frac{N}{6} + \frac{1}{6}$	$\frac{1}{6}$
23	$\frac{1}{4}$	$\frac{1}{4}$
24	$\frac{1}{4}$	$\frac{1}{4}$

Table 2: List of the 27-plet divergent counterterms D_i at order p^4 for N generic. Notation as in table 1.

In the 27-plet sector all the unquenched chiral invariants D_i survive to the quenched approximation. The operators D_i, $i = 1, 2, 4, 8, 13, 14, 15, 16, 17, 18, 21, 22$ loose their linear flavour number dependent contribution, D_{12} looses the $1/N$ term due to the presence of the dynamical singlet field, while getting a contribution proportional to α. D_{10} gets a contribution from $\tilde{v}_{27}$. Counterterms $D_1, \dots D_7$ contribute to $K \to 3\pi$ decays, $D_8, \dots D_{12}$ to $K \to 2\pi$, 3π decays, $D_{13}, \dots D_{16}$ to radiative K decays and $D_{17}, \dots D_{24}$ to processes with an external W gauge boson.

5. Applications to weak observables

In this section we use the results obtained through the calculation of the full and quenched weak generating functional to one loop to estimate in a few cases how the quenched approximation modifies the contribution coming from chiral logarithms to nonleptonic weak matrix elements. We focus on B_K in the $\Delta S = 2$ sector and $K \to \pi\pi$ matrix elements in the $\Delta S = 1$ sector. The B_K parameter has been extensively investigated in the literature both in the unquenched case (see e.g. refs. [16, 17, 18, 19]) and in the quenched approximation (see e.g. refs. [19, 16, 20] and refs. [21, 22] for recent lattice determinations). Recent attempts to compute unquenched (really partially quenched) B_K on the lattice can be found in refs. [23, 24, 25]. The same interest has been devoted to the formulation of analytic approaches to $K \to \pi\pi$ decays in the full theory, see e.g. [14, 15, 26, 27, 28, 29]. An attempt to fix the order p^4 counterterms in $K \to \pi\pi$ decays from the off-shell $K\pi$, $K\eta$ weak transitions has been done in ref. [15]. In the quenched approximation, a first analysis of the $\Delta I = 3/2$ decay $K^+ \to \pi^+\pi^0$ can be found in ref. [16], where finite volume effects on the lattice are also investigated. The most recent numerical determination of $K^+ \to \pi^+\pi^0$ amplitude on the lattice and in the quenched approximation is reported in ref. [30]. In what follows we shall concentrate on the comparison of chiral logarithms that contribute to the various weak quantities in the full and in the quenched theory. Although some of the results are well known, as in the case of B_K, the structure of the counterterms and their flavour number dependence will be clear in this context.

5.1 The B_K parameter

The B_K parameter is defined in terms of $K_0\bar{K}_0$ amplitudes as follows

$$\mathcal{M}_K = \left\langle K_0|O_K(x)|\bar{K}_0\right\rangle = B_K(\mu)\,\mathcal{M}_{vac}\,. \tag{5.1}$$

$O_K(x)$ is the effective four-quark operator $O_K(x) = (\bar{s}(x)\gamma_\mu(1-\gamma_5)d(x))(\bar{s}(x)\gamma^\mu(1-\gamma_5)d(x))$, where summation over colours is understood within brackets, and $\mathcal{M}_{vac}$ is the result of the vacuum saturation approximation

$$\mathcal{M}_{vac} = \frac{8}{3}\left\langle K_0|\bar{s}\gamma_\mu(1-\gamma_5)d|0\right\rangle\left\langle 0|\bar{s}\gamma^\mu(1-\gamma_5)d|\bar{K}_0\right\rangle. \tag{5.2}$$

The scale dependence of $B_K(\mu)$ is due to the fact that the effective four-quark operator $O_K(x)$ has an anomalous dimension. The scale independent quantity is the physical matrix element $\langle K_0|\mathcal{H}_{\Delta S=2}|\bar{K}_0\rangle$, where the effective Hamiltonian $\mathcal{H}_{\Delta S=2}$ can be written as $\mathcal{H}_{\Delta S=2} = -C^{\Delta S=2}\hat{O}_K(x)$, with the constant $C^{\Delta S=2}$ defined in eq. (2.15), while the matrix element of the four-quark operator $\hat{O}_K(x)$ is now scale invariant and defines the renormalization group invariant $\hat{B}_K = B_K(\mu)\alpha_S(\mu)^{a_+}$ [31]. The realization of $\hat{O}_K(x)$ in terms of low energy degrees of freedom is contained in the effective

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Lagrangian of eq. (2.5) that gives $\hat{O}_K(x) = G_{27}F^4\langle\Delta_{32}u_\mu\rangle\langle\Delta_{32}u_\mu\rangle$. At order p^2 in the chiral expansion one obtains

$$\mathcal{M}_K = 4G_{27}F^2m^2\,, \qquad \mathcal{M}_{vac} = \frac{16}{3}F^2m^2\,, \tag{5.3}$$

where F and m^2 are the bare kaon decay constant and squared mass and

$$\hat{B}_K = \frac{3}{4}G_{27}\,, \tag{5.4}$$

with $G_{27} = 1$ in the large-N_c limit.[5] The same result is valid in the quenched approximation where the bare parameter G_{27}, the squared meson mass m^2 and the meson decay constant F are replaced by the quenched ones. At order p^4 in the chiral expansion the vacuum saturation amplitude gets renormalized so that

$$\mathcal{M}_{vac} = \frac{16}{3}F_K^2m_K^2\,. \tag{5.5}$$

The full amplitude $\mathcal{M}_K$ receives contributions from the strong and weak sectors. A detailed analysis at order p^4 in ChPT can be found in ref. [18]. Within the $1/N_c$ expansion one can distinguish the factorizable (i.e. the non zero ones in the large N_c limit) and the non-factorizable diagrams at order p^4. The factorizable contributions provide the renormalization of masses and decay constants. The non-factorizable contributions are the only ones relevant to B_K. We write B_K as the sum $B_K = B_K|_{ctr} + B_K|_{logs}$, where the first term contains the analytic contributions coming from the effective Lagrangian up to order p^4 and the second term contains the non-analytic corrections coming from the one-loop diagrams. The generating functional as it was derived in the previous sections gives in one step the divergent part of the analytic contribution to any hadronic weak matrix element; in other words it gives the coefficients of the chiral logarithms that contribute at one loop to any hadronic weak matrix element, both in the full theory and in the quenched theory. We limit the analysis to $B_K|_{ctr}$ with the purpose of illustrating how the quenched approximation modifies the coefficients of the chiral logarithms in the degenerate mass case. At order p^4 in the full theory $B_K|_{ctr}$ reads as follows

$$B_K|_{ctr} = \frac{3}{4}G_{27}\left\{1 + \frac{m_K^2}{F^2}\left(A+B+C\right)\right\}, \tag{5.6}$$

with

$$\begin{aligned} A &= -16\,d_{12}\left(1 - \frac{m_\pi^2}{m_K^2}\right)^2\,, \\ B &= 8(d_{10} - 2L_4)\frac{2B_0\langle\mathcal{M}\rangle}{m_K^2} + 16(d_8 - L_5)\,, \\ C &= -16\,d_f\,, \end{aligned} \tag{5.7}$$

[5] At order p^2 in ChPT the same parameter G_{27} relates B_K to the $\Delta I = 3/2$ amplitude $K^+ \to \pi^+\pi^0$ [17]. One loop corrections due to $SU(3)$ breaking were derived in [32].

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where d_8, d_{10}, d_{12} are the weak $\Delta S = 2$ counterterms defined in eq. (4.6) and L_4, L_5 are the strong counterterms as defined in eq. (4.7) (see also ref. [10] for their definition in the full and quenched case). The contribution from C is due to one finite counterterm $q\langle\Delta\chi_-\rangle\langle\Delta\chi_-\rangle$ with coefficient d_f.

In the degenerate quark mass case (i.e. $\mathcal{M} = m_q\mathbf{1}$) A is zero, while the first coefficient in B develops an extra linear flavour number dependence. Notice that the combinations $d_{10} - 2L_4$ and $d_8 - L_5$ in B come from the residual non-factorizable part of the total contribution from counterterms D_{10} and D_8, while their factorizable part goes into the renormalization of F_K in the full $\mathcal{M}_K$ matrix element. In addition, the flavour number dependence of D_8 and D_{10} is exclusively contained in their factorizable part. It is now a simple exercise to show that in the degenerate quark mass case the chiral logarithms contributing to B_K at one loop are the same in the full theory and in the quenched theory (i.e. no flavour number dependence is produced in the full theory for degenerate quark masses). In the quenched degenerate mass case the operator D_{10} does not contribute anymore to B_K so that at order p^4 one gets

$$B^q_K|_{ctr} = \frac{3}{4}G^q_{27}\left\{1 + \frac{M_K^2}{F^{q2}}\left(A^q + B^q + C^q\right)\right\}, \tag{5.8}$$

with

$$\begin{aligned} A^q &= 0\,, \\ B^q &= 16(d_8^q - L_5^q)\,, \\ C^q &= -16\, d_f^q\,, \end{aligned} \tag{5.9}$$

and the weak coupling, mass and decay constant are the quenched ones. Notice also that L_5^q in the quenched case becomes finite.

The renormalized $B_K|_{ctr}$ can be obtained from eq. (5.6) through the substitution $L_i \to \frac{\Gamma_i}{32\pi^2}\bar{L}_i$ and $d_i \to \frac{\delta_i}{32\pi^2}\bar{d}_i$ (with the exception of d_f), where the barred constants are the scale independent constants defined in eq. (4.9).

The renormalized $B^q_K|_{ctr}$ can analogously be written in terms of $\bar{L}_i^q$ and $\bar{d}_i^q$ through the same substitution (with the exception of L_5^q) in eq. (5.8). The scale independent constants carry the chiral logarithm, e.g. $\bar{d}_i = -\ln M^2 + \cdots$. By inserting in the renormalized expressions of $B_K|_{ctr}$ and $B^q_K|_{ctr}$ the values of the coefficients δ_i and δ_i^q as given in table 2 and using $\Gamma_4 = 1/8$, $\Gamma_5 = N/8$ for the coefficients of the divergences in L_4 and L_5 we get the contribution to B_K coming from the chiral logarithms in the full and quenched theory

$$\begin{aligned} B_K|_{ctr} &= \frac{3}{4}G_{27}\left\{1 - 6\,\frac{m_K^2}{32\pi^2F^2}\ln\frac{M^2}{\mu^2} + \cdots\right\}, \\ B^q_K|_{ctr} &= \frac{3}{4}G^q_{27}\left\{1 - 6\,\frac{M_K^2}{32\pi^2F^{q2}}\ln\frac{M^2}{\mu^2} + \cdots\right\}, \end{aligned} \tag{5.10}$$

where the renormalization scale dependence cancels in the sum. eq. (5.10) proves the flavour independence of the next-to-leading contribution to B_K in the full theory and the consequent equality of the coefficients of chiral logarithms in the full and quenched case for degenerate quark masses.[6]

5.2 $K \to \pi\pi$ matrix elements

The analysis of $K \to \pi\pi$ matrix elements in the full theory, using the effective weak chiral Lagrangian at order p^4 has been done in ref. [15]. Using those formulas and the quenched counterpart of the divergent counterterms derived in the previous sections, we can produce a few quantitative estimates of quenching effects on the coefficients of chiral logarithms that contribute to $K \to \pi\pi$ amplitudes. We work in the infinite volume limit for illustrative purpose, while we defer to future work the analysis of aspects more closely related to an actual lattice simulation. We consider $K \to \pi\pi$ matrix elements with $\Delta I = 1/2$ and $\Delta I = 3/2$ in the full theory at one loop and derive the modifications induced by quenching in the coefficients of chiral logarithms for degenerate light quark masses $m_u = m_d \equiv \hat{m}, \hat{m} = m_s$ (i.e. $m_K = m_\pi$). The amplitudes in the full theory are first computed in the case where no dynamical singlet component is present and for non degenerate quark masses. Next, we consider the two limits $m_K = m_\pi$ and $m_\pi = 0$. In the first case, $m_K = m_\pi$, the explicit flavour number dependence of the full amplitudes is shown. Once the flavour number dependence of the full amplitudes is known in the degenerate mass case, it is immediate to derive the corresponding quenched expression for degenerate masses. Each coefficient of the chiral logarithms in the full amplitudes is replaced by its quenched value according to tables 1 and 2. This in practice amounts to eliminate the flavour number dependence of the full amplitudes and to add new contributions (proportional to m_0^2, α, v_i, $\tilde{v}_i$) coming from the anomalous singlet sector.

We decompose the $K \to \pi\pi$ matrix elements into definite isospin invariant amplitudes as follows

$$\begin{aligned}
A\left[K_S \to \pi^0\pi^0\right] &\equiv \sqrt{\frac{2}{3}}A_0 - \frac{2}{\sqrt{3}}A_2\,,\\
A\left[K_S \to \pi^+\pi^-\right] &\equiv \sqrt{\frac{2}{3}}A_0 + \frac{1}{\sqrt{3}}A_2\,,\\
A\left[K^+ \to \pi^+\pi^0\right] &\equiv \frac{\sqrt{3}}{2}A_2\,,
\end{aligned} \tag{5.11}$$

where $K_S \simeq K_1^0 + \varepsilon\, K_2^0$, $K_{1(2)}^0 \equiv (K^0 - (+)\overline{K^0})/\sqrt{2}$, CP $K_{1(2)}^0 = +(-)K_{1(2)}^0$ and we set $\varepsilon = 0$ since CP violation is small. The isospin 1/2 amplitude can be written as

[6] No contribution from the anomalous singlet sector is present in the quenched degenerate mass case.

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follows

$$A_0 \equiv -ia_0\, e^{i\delta_0} \tag{5.12}$$

and the analogous for the isospin 3/2 amplitude

$$A_2 \equiv -ia_2\, e^{i\delta_2}\,, \tag{5.13}$$

where $\delta_{0,2}$ are final state interaction phases. At order p^2 we get

$$\begin{aligned} \Im m A_0 \equiv \Im m(A_0^8 + A_0^{27}) &= -\left[g_8 + \frac{1}{9}g_{27}\right]\frac{\sqrt{6}}{F^3}\,(m_K^2 - m_\pi^2)\,, \\ \Im m A_2 &= -\,g_{27}\,\frac{10\sqrt{3}}{9F^3}\,(m_K^2 - m_\pi^2)\,, \end{aligned} \tag{5.14}$$

where we use $F = 93$ MeV and

$$\delta_0 = \delta_2 = 0\,. \tag{5.15}$$

In the quenched approximation each parameter in eq. (5.14) (i.e. masses, weak couplings and decay constant) has to be replaced by its quenched counterpart. At order p^4 in the chiral expansion we define the full generic amplitude $\Im m A_i$ as the sum $\Im m A_i|_{ctr} + \Im m A_i|_{logs}$, where the first term contains the analytic contributions coming from the effective Lagrangian up to order p^4 and the second term contains the non-analytic corrections coming from the one-loop diagrams. Both contributions have been derived for the physical amplitudes in ref. [15]. In the following we generalize the analytic part of the amplitudes to a generic number of flavours; one can then obtain its quenched approximation by replacing each divergence in the full amplitude by its quenched value and setting to zero any residual flavour number dependence (i.e. the one coming from $\langle\mathcal{M}\rangle$ contributions at degenerate quark masses). This procedure is the most immediate to obtain the coefficients of chiral logarithms in the full and quenched theory and in the degenerate mass case.

For the octet physical amplitude in the full theory, with $\hat{m} \neq m_s$, we get

$$\begin{aligned} \Im m A_0^8|_{ctr} = &-g_8 \frac{\sqrt{6}}{F_K F_\pi^2}(m_K^2 - m_\pi^2)\Big\{1 + \frac{2}{F^2}\Big[4m_\pi^2\Big(2w_5 + 4w_7 - 2w_{10} - \\ &-4w_{11} - 2w_{12} + w_{38}\Big) + 4m_K^2\Big(w_5 - 2w_7 + w_9\Big) + 8B_0\langle\mathcal{M}\rangle w_8\Big]\Big\} - \\ &-g_8' \frac{8\sqrt{6}}{F_K F_\pi^2}\frac{(m_K^2 - m_\pi^2)}{F^2}\Big[m_\pi^2\Big(-4L_4 - L_5 + 8L_6 + 4L_8\Big) + 2m_K^2 L_4\Big], \end{aligned} \tag{5.16}$$

where the meson decays constants F_K and F_π in the leading order p^2 amplitude are the renormalized ones; their expression at order p^4 is given in refs. [13, 15]; their analytic contribution at order p^4 for a generic N gives the following result

$$F_K F_\pi^2|_{ctr} = F^3\left[1 + \frac{24B_0\langle\mathcal{M}\rangle}{F^2}L_4 + \frac{4}{F^2}\left(m_K^2 + 2m_\pi^2\right)L_5\right]. \tag{5.17}$$

Figure 1: Tree level diagrams that contribute to $K \to \pi\pi$ matrix elements at order p^4. The crossed box is the insertion of a weak counterterm (octet w_i or 27-plet d_i). The crossed dot is the insertion of a strong counterterm L_i. The box is an order p^2 weak vertex (g_8, g_8', g_{27}). The dot is an order p^2 strong vertex. Diagram *(d)* is a wave function renormalization diagram for each external leg.

In figure 1 we list the tree level diagrams which give contribution to all $K \to \pi\pi$ amplitudes up to order p^4 in the chiral expansion. The first two lines of eq. (5.16) come from diagrams (a) and (c) of figure 1. Diagram (c) gives contribution only to w_{11}. The third line comes from diagram (b), while diagram (d) for each external line gives the renormalization of $F_K F_\pi^2$ in the leading order p^2 amplitude. Meson masses m_K and m_π are always the renormalized physical ones because they come from factors p_K^2, p_π^2, where we take the external momenta on shell.

The expression in eq. (5.16) is valid for a generic number of flavours N. In this respect it differs from the one in ref. [15]. Also, our choice of the octet basis slightly differs from the one adopted in there, while it coincides with the one chosen in ref. [5] when counterterm W_{38} is eliminated in the three flavours case. All counterterms in eq. (5.16) are divergent. In the degenerate quark mass limit the counterterm W_8 induces a linear flavour number dependence due to the term $2B_0\langle\mathcal{M}\rangle \to N m_\pi^2$. Besides, W_{11} yields a linear flavour number dependence in the degenerate mass case, being proportional to $\langle\chi_+\rangle$. However this contribution drops out in the sum of diagrams *(a)* and *(c)* depicted in fig. 1. The strong counterterms with coefficients L_4 and L_5 again produce a linear flavour number dependence in the wave function renormalization diagrams of the type shown in *(d)* of figure 1, which is fully reabsorbed in the renormalization of the product $F_K F_\pi^2$ of meson decay constants.

For instructive purpose we derive the flavour number dependence of the coefficients of chiral logarithms that enter the octet amplitude at order p^4 and in the degenerate quark mass case. To this aim we first obtain the fully renormalized analytic contribution at order p^4 by performing the substitution $w_i \to \frac{\nu_i}{32\pi^2}\bar{w}_i$ and $L_i \to \frac{\Gamma_i}{32\pi^2}\bar{L}_i$ in eq. (5.16), where $\bar{L}_i$, $\bar{w}_i$ are the scale independent constants defined in eq. (4.9). Then we use the fact that $\bar{w}_i = -\ln m_K^2 + \ldots$, $\bar{L}_i = -\ln m_K^2 + \ldots$ to derive the coefficients of chiral logarithms. Their contribution is given by the following ratio

$$\frac{\Im m A_0^8|_{ctr}}{\Im m A_0^8|_{tree}}|_{\hat{m}=m_s} = 1 - \left(\frac{1}{2} - \frac{6}{N} + \frac{8}{N^2}\right)\frac{m_K^2}{16\pi^2 F^2}\ln\frac{m_K^2}{\mu^2} + \cdots\,, \tag{5.18}$$

where we expressed everything in terms of m_K^2 and the μ dependence cancels in the complete amplitude. $\Im m A_0^8|_{tree}$ is the order p^2 bare amplitude of eq. (5.14), also obtained from eq. (5.16) when the product $F_K F_\pi^2$ has been converted to the unrenormalized F^3 through the relation (5.17); the divergent contribution to (5.17) is obtained using $\Gamma_4 = 1/8$, $\Gamma_5 = N/8$ for the coefficients of the divergence in L_4 and L_5. Notice that the chiral logarithm coming from the strong counterterms part $g_8' \cdot L_i$ in eq. (5.16) exactly cancels the one proportional to g_8' coming from the weak counterterms w_i's. For $N = 3$ the coefficient of the chiral logarithm in eq. (5.18) is (11/18), which becomes $-(35/9)$ when $F^3 \to F_K F_\pi^2$ is replaced in the tree level amplitude.[7]

The analytic contributions to the 27-plet physical amplitudes up to order p^4 are as follows:

$$\Im m A_0^{27}|_{ctr} = -g_{27}\frac{\sqrt{6}}{9F_K F_\pi^2}\left(m_K^2 - m_\pi^2\right)\left\{1 + \frac{1}{F^2}\left[8m_\pi^2\left(2d_8 - 2d_f + 6d_9 - 6d_{12}\right) + 4m_K^2\left(d_8 - 9d_9 + d_{11}\right) + 16B_0\langle\mathcal{M}\rangle d_{10}\right]\right\} \tag{5.19}$$

$$\Im m A_2|_{ctr} = -g_{27}\frac{10\sqrt{3}}{9F_K F_\pi^2}\left(m_K^2 - m_\pi^2\right)\left\{1 + \frac{1}{F^2}\left[16m_\pi^2(d_8 - d_f) + 4m_K^2\left(d_8 + d_{11}\right) + 16B_0\langle\mathcal{M}\rangle d_{10}\right]\right\}. \tag{5.20}$$

In the 27-plet case there is one finite counterterm with coefficient d_f which gives contribution. This counterterm is $q\langle\Delta\chi_-\rangle\langle\Delta\chi_-\rangle$ and it does not produce any flavour number dependence. As in the octet case, counterterm D_{10} yields a linear flavour number dependence in the degenerate mass case. Again, the renormalized analytic contribution is obtained by performing the substitution $d_i \to \frac{\delta_i}{32\pi^2}\bar{d}_i$ (with the exception of the finite term d_f) in eqs. (5.19) and (5.20). The flavour number dependence of the coefficients of chiral logarithms in the 27-plet case is now given by the following ratios

$$\frac{\Im m A_0^{27}|_{ctr}}{\Im m A_0^{27}|_{tree}}|_{\hat{m}=m_s} = 1 + \left(-\frac{3}{4}N - 4 + \frac{6}{N}\right)\frac{m_K^2}{16\pi^2 F^2}\ln\frac{m_K^2}{\mu^2} + \cdots, \tag{5.21}$$

which gives $-(17/4)$ for the coefficient of the chiral logarithm with $N = 3$ (it goes to $-(35/4)$ when $F^3 \to F_K F_\pi^2$) and

$$\frac{\Im m A_2|_{ctr}}{\Im m A_2|_{tree}}|_{\hat{m}=m_s} = 1 + \left(-\frac{3}{4}N - \frac{13}{4}\right)\frac{m_K^2}{16\pi^2 F^2}\ln\frac{m_K^2}{\mu^2} + \cdots, \tag{5.22}$$

which gives $-(11/2)$ for the coefficient of the chiral logarithm with $N = 3$ (it goes to -10 when $F^3 \to F_K F_\pi^2$). The non degenerate case with $m_\pi = 0$ moves the coefficients of chiral logarithms to $-(15/2)$ and $-(3/4)$ in eq. (5.21) and (5.22) respectively.

[7] Notice that the coefficient of the chiral logarithm in eq. (5.18) becomes $-(5/4)$ in the non degenerate case for $m_\pi = 0$. This can be derived from eqs. (5.16) and (5.17) by setting $m_\pi = 0$.

The quenched version of the analytic contribution to the $\Im m A_i$ amplitudes at order p^4 and in the degenerate mass case can be easily derived by replacing the quenched values for the coefficients of the divergences ν_i, δ_i, Γ_i, as given in ref. [10] and tables 1, 2 and dropping the residual flavour number dependence due to the $\langle \mathcal{M} \rangle$ for degenerate masses. Again, we define the limit of equal masses of the counterterm over tree amplitude ratio and we define as M the degenerate meson mass in the quenched amplitudes. The quenched octet ratio is given by:

$$\frac{\Im m A_0^8|^Q_{ctr}}{\Im m A_0^8|^Q_{tree}}|_{\hat{m}=m_s} = 1 + \frac{2M^2}{F^2}\Big(12w_5^q + 8w_7^q + 4w_9^q - 8w_{10}^q - 16w_{11}^q - 8w_{12}^q + 4w_{38}^q\Big) + \frac{g_8'}{g_8}\frac{8M^2}{F^2}\Big(- 2L_4^q - L_5^q + 8L_6^q + 4L_8^q\Big)\,, \tag{5.23}$$

where the weak order p^2 couplings and decay constant are the quenched ones. Notice that in the quenched case the product of meson decay constants in the leading order amplitude is only renormalized at one loop by a finite counterterm [10]. It is important to note that the weak mass term contribution from g_8' never appears in the leading order amplitude, but it gives a residual contribution of the type $g_8' \cdot L_i^q$ to (5.23). As we saw in section (3.3) the quenched chiral logarithms generated in the weak interaction sector can be formally reabsorbed into a redefinition of the weak coupling $g_8' \to g_{8R}'$. In addition, the anomalous behaviour of the singlet sector in the quenched approximation yields a modification in the power counting due to the presence of a new dimensionful expansion parameter m_0^2, the singlet squared mass. For this reason, quenched ChPT becomes a double expansion in powers of the standard chiral parameter $\epsilon \equiv M^2/16\pi^2F^2$ and the singlet parameter $\delta \equiv m_0^2/16N_c\pi^2F^2$ (which induces a $1/N_c$ expansion). eq. (5.23) shows that in the degenerate mass case quenched chiral logarithms cannot appear at one loop, while the one obtained from the rescaling of $g_8' \to g_{8R}'$ in eq. (5.23) is already a two loop effect and of order $\epsilon \cdot \delta$ in the combined chiral and $1/N_c$ expansion.[8]

The 27-plet quenched ratios are as follows:

$$\frac{\Im m A_0^{27}|^Q_{ctr}}{\Im m A_0^{27}|^Q_{tree}}|_{\hat{m}=m_s} = 1 + \frac{M^2}{F^2}\left(20d_8^q - 16d_f^q + 12d_9^q + 4d_{11}^q - 48d_{12}^q\right), \tag{5.24}$$

$$\frac{\Im m A_2|^Q_{ctr}}{\Im m A_2|^Q_{tree}}|_{\hat{m}=m_s} = 1 + \frac{M^2}{F^2}\left(20d_8^q - 16d_f^q + 4d_{11}^q\right). \tag{5.25}$$

As it happens in the full theory only one finite counterterm $q\langle \Delta\chi_- \rangle \langle \Delta\chi_- \rangle$ with coefficient d_f^q contributes to the 27-plet amplitudes.

The contribution from the chiral logarithms to the quenched amplitudes can be derived with the same procedure adopted in the full case, i.e. through the derivation

[8] Notice however that the order p^2 amplitude always contains the fully renormalized meson masses.

of the renormalized amplitudes in terms of the scale independent constants $\bar{w}_i^q$, $\bar{d}_i^q$, $\bar{L}_i^q$ and using the quenched values of the coefficients ν_i, δ_i as given in tables 1 and 2. For the coefficients of the divergence in the strong $\bar{L}_i^q$ we use $\Gamma_4^q = 1/8 - v_1/2$, $\Gamma_6^q = 1/16 - v_2/2 + \alpha^2/72$, $\Gamma_8^q = -\alpha/12$, as derived in ref. [10]. The result reads as follows:

$$\frac{\Im m A_0^8|_{ctr}^Q}{\Im m A_0^8|_{tree}^Q}|_{\hat{m}=m_s} = 1 - \epsilon \left[\frac{1}{2} + 8v_1 - 16v_2 + \frac{4}{9}\alpha^2 - 2\alpha + 8\frac{g_8'}{g_8}\left(-v_2 + \frac{\tilde{v}_5}{g_8'}\right) + \right.$$
$$\left. + 2\frac{\bar{g}_8}{g_8}\left(1 - \frac{2}{3}\alpha\right)\right] \ln\frac{M^2}{\mu^2} + \cdots, \tag{5.26}$$

$$\frac{\Im m A_0^{27}|_{ctr}^Q}{\Im m A_0^{27}|_{tree}^Q}|_{\hat{m}=m_s} = 1 - 4\left(1 - \frac{1}{2}\alpha\right)\epsilon \ln\frac{M^2}{\mu^2} + \cdots, \tag{5.27}$$

$$\frac{\Im m A_2|_{ctr}^Q}{\Im m A_2|_{tree}^Q}|_{\hat{m}=m_s} = 1 - \frac{13}{4}\epsilon \ln\frac{M^2}{\mu^2} + \cdots, \tag{5.28}$$

where we defined $\epsilon = M^2/16\pi^2F^2$. While in the $\Delta I = 3/2$ amplitude A_2 no trace of the singlet anomalous sector is present through chiral logarithms proportional to the singlet parameters α, v_i and $\tilde{v}_i$, this is not the case for the $\Delta I = 1/2$ amplitudes also in the degenerate quark mass limit.

In figure 2 we clarify the origin of the one loop contributions to eq. (5.27) that are proportional to the strong singlet potentials v_1, v_2 and the weak singlet potential $\tilde{v}_5$. No contribution proportional to $\tilde{v}_8$ nor $\tilde{v}_{27}$ is allowed in this case. The total contribution to eq. (5.27) and proportional to $g_8' \cdot v_1$ is zero due to the exact cancellation of loops (b) and (c) in figure 2. For v_2 the extra one loop contribution shown in (d) of figure 2 gives the net divergence in A_0^8 in the quenched case. Contributions proportional to α and α^2 are the usual ones coming from one or two insertions of the double pole term of the singlet propagator in a one loop diagram (no more than one insertion per singlet line).

In table 3 we compare the coefficients of chiral logarithms for the three amplitudes in the full theory and in the quenched approximation. For the full amplitudes we consider two extreme mass configurations in the one loop corrections: a) degenerate masses, i.e. $m_K = m_\pi$ and b) $m_\pi = 0$. The numerical analysis of the physical

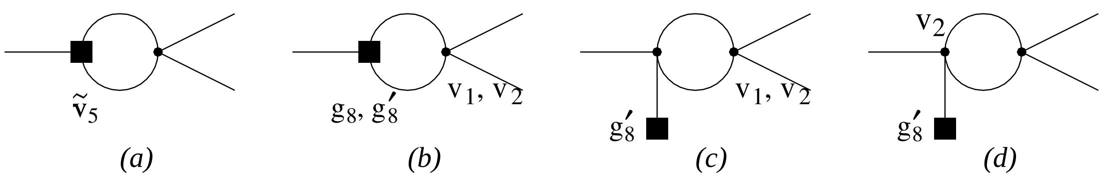

Figure 2: One loop contributions to $K \to \pi\pi$ matrix elements with a single insertion of a weak $\tilde{v}_i$ or a strong v_i vertex in the quenched approximation. A singlet field is always running in the loop. Notation is as in figure 1.

	Full ($m_K = m_\pi$)	Full ($m_\pi = 0$)	Quenched
A_0^8	$\frac{11}{18}$	$-\frac{5}{4}$	$-\left(\frac{1}{2} + \frac{4}{9}\alpha^2 - 2\alpha\right) + v_i, \tilde{v}_i, \bar{g}_8, g_8'$
A_0^{27}	$-\frac{17}{4}$	$-\frac{15}{2}$	$-4\left(1 - \frac{\alpha}{2}\right)$
A_2	$-\frac{11}{2}$	$-\frac{3}{4}$	$-\frac{13}{4}$

Table 3: The coefficient of $m^2/(16\pi^2 F^2)\ln(m^2/\mu^2)$ (with $m^2 = m_K^2$ (Full), $m^2 = M^2$ (Quenched)) for the full amplitude with $m_K = m_\pi$ (second column), $m_\pi = 0$ (third column) and for the quenched amplitude (fourth column).

non degenerate mass case together with the comparison with *unphysical* choices of the matrix elements on the lattice will be given in ref. [8].

Notice that the coefficient of the chiral logarithm in the quenched octet amplitude cannot be determined because of its dependence upon unknown parameters of the singlet sector.

The coefficients of chiral logarithms in the full 27-plet amplitudes $\Im m A_0^{27}$ and $\Im m A_2$ are quite large in the degenerate mass limit. In addition, by the comparison of second and third column in table 3 one can conclude that all the coefficients in the full amplitudes are extremely sensitive to the variation of masses. Going from the degenerate mass case to the limit $m_\pi = 0$ the coefficient of the $\Delta I = 3/2$ amplitude decreases in absolute value by a large amount (from -5.5 to -0.75, $\sim$ 86%), while the one of the $\Delta I = 1/2$ amplitude increases by $\sim$ 76% (from -4.25 to -7.5) in the 27-plet case. The coefficient of the octet case changes sign going from 0.6 to -1.25. Going from the case $m_K = m_\pi$ to the case $m_\pi = 0$ we notice a drastic increase of the corrections to the $\Delta I = 1/2$ amplitudes, in contrast to a sizable decrease of the corrections to the $\Delta I = 3/2$ amplitude. The values of the coefficients at $m_\pi = 0$ clearly produce an enhancement of the $\Delta I = 1/2$ amplitude. In ref. [8] a more complete analysis of the physical case will be given.

Setting $\alpha \simeq 0$ and disregarding for now the unknown contributions to the octet amplitude in the quenched case we find the following pattern going from the full degenerate mass case to the quenched one. Quenching reduces from 0.6 to -0.5 the coefficient of the chiral logarithm in the octet amplitude A_0^8. Notice however that the quenched result is very sensitive to the numerical value of the parameter α. At present, no precise measurement of α is available on the lattice.

The coefficient in A_0^{27} is only reduced by about 6% in absolute value (from -4.25 to -4). Again, a value of α different from zero affects the result. For example, $\alpha = 0.6$ would give a reduction of about 34% (from -4.25 to -2.8). Comparing instead with the $m_\pi = 0$ full case, quenching, with $\alpha = 0$, reduces the coefficient in A_0^{27} by about 47% (from -7.5 to -4). In the case of A_2, comparing with the full degenerate mass case, the coefficient of the chiral logarithm is reduced by about 41% (from -5.5 to

-3.25). No dependence upon α is produced. The pattern is opposite for A_2 when we compare the quenched amplitude to the $m_\pi = 0$ limit of the full amplitude. In this case quenching increases the coefficient in absolute value from -0.75 to -3.25, producing an enhancement of the $\Delta I = 3/2$ amplitude.

This analysis shows that in most of the cases we have considered here, quenching tends to produce sizable modifications of the coefficients of the chiral logarithms in the $\Delta I = 3/2$ and $\Delta I = 1/2$ amplitudes. However, the knowledge of the parameter α is essential in order to improve the determination of the quenched $\Delta I = 1/2$ amplitudes. The comparison with the $m_\pi = 0$ limit of the full amplitudes, expected to be the most approximate to the physical value, has shown that the modification induced by quenching (with $\alpha = 0$ and degenerate quark masses) follows a pattern that tends to suppress the $\Delta I = 1/2$ dominance.

6. Summary and conclusions

We presented the derivation of the ultraviolet divergences of the generating functional for hadronic weak interactions at one loop in the full theory and in the quenched approximation. Weak interactions with $\Delta S = 1$, $\Delta I = 1/2$ and $\Delta I = 3/2$, and $\Delta S = 2$ have been considered. The aim was double. Within the full theory we added a few results to the previous analyses: we derived the modifications induced by the inclusion of a dynamical singlet field, and constructed a minimal basis of divergent counterterms at order p^4, both in the octet and 27-plet sector. The whole derivation is done for a generic number of flavours N. The list of ultraviolet divergent counterterms and the coefficients of the divergences in the full theory with N generic are given in tables 1 and 2. The generating functional in the full theory and for $N = 3$ was already derived in ref. [4], while a minimal basis for the octet sector was optimized in ref. [5]; there, the modifications induced by a singlet dynamical field were anticipated (see appendix A of ref. [5]). Here, the modifications induced in the generating functional by the singlet dynamical field are analyzed in more detail (see section (3.4) for a summary of the results).

Within the quenched approximation the aim was to develop a systematic tool for treating the one loop corrections of quenched ChPT [11] in the presence of weak interactions. The method used here (i.e. the generating functional approach) was already proposed in ref. [10] in the case of the quenched approximation of ChPT for strong interactions, while it was first introduced in standard ChPT in refs. [12, 13]. The presence of weak interactions does not spoil any of the needed properties of the generating functional, while we perform an expansion in powers of the coupling G_F.

The cancellation of the flavour number dependence induced by the quenching procedure is verified within the present approach. The linear flavour number dependence is cancelled by the fermionic ghost determinant, while the cancellation of the inverse powers $1/N$, $1/N^2$ is due to the appearance of a dynamical singlet field. The modifications induced by quenching are summarized in tables 1 and 2. Quenching largely affects the structure of the divergences in the octet sector of the theory, where the number of divergent counterterms goes from 25 to 10, while all the divergences in the 27-plet sector remain, loosing their flavour number dependence.

An interesting feature of the quenched approximation in the presence of weak interactions is the appearance of new *quenched chiral logarithms* (i.e. of the type $m_0^2 \log m_\pi^2$, where m_0 is the singlet mass), in addition to the ones generated by quenched strong interactions. In ref. [10] it was shown that quenched chiral logarithms appearing at one loop of the strong interactions amount to a redefinition of the B_0 parameter which measures the quark condensate in ChPT. Here we have shown that the quenched chiral logarithms induced by weak interactions can be accounted for via a redefinition of the weak mass term coupling g_8' of the leading order weak Lagrangian.

Once the ultraviolet divergences of the weak generating functional to one loop are known in the full and in the quenched theory and for degenerate quark masses, all the coefficients of chiral logarithms are known in both theories, for any nonleptonic weak matrix element. As an application of this result, we have considered B_K and $K \to \pi\pi$ matrix elements to one loop, in order to extract the modifications induced by quenching in the coefficient of chiral logarithms. At this stage the analysis is done at infinite volume. The well known result for B_K [19] is found and the structure of the divergent counterterms is clarified for this quantity. For the contribution of chiral logarithms to $K \to \pi\pi$ matrix elements in the full theory we used the results derived in ref. [15] and generalized them to a generic number of flavours N. The quenched expression of each matrix element is also given for degenerate light quark masses. In ref. [8] we shall report a more detailed numerical comparison of the physical amplitudes with lattice amplitudes, i.e. where an *unphysical* choice of the kinematics is done with or without the quenched approximation.

Main result of the numerical analysis is that, in most of the cases considered here, the quenched approximation in the degenerate mass case tends to produce sizable modifications of the coefficients of chiral logarithms in the $\Delta I = 1/2$ and $\Delta I = 3/2$ amplitudes (see table 3). However, the knowledge of the singlet parameter α is essential in order to improve the determination of the quenched $\Delta I = 1/2$ amplitudes.

Also, the pattern of the corrections in respect to the $\Delta I = 1/2$ rule goes as follows. The contribution from chiral logarithms in the full (unquenched) theory and for $m_\pi = 0$ clearly produces an enhancement of the $\Delta I = 1/2$ amplitude. Instead, the comparison of the values of chiral logarithms in the quenched amplitudes with the $m_\pi = 0$ values of the chiral logarithms in the full amplitudes shows that the modifications induced by quenching follow a pattern that tends to suppress the $\Delta I = 1/2$ dominance.

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A. List of weak operators

We define here the weak operators that carry the ultraviolet divergences of the weak generating functional to one loop at order G_F. With W_i we define the octet operators which are nonzero in the absence of the singlet field (for them we follow the same ordering as given in ref. [5]). D_i, $i = 1, \ldots 24$ are 27-plet operators which are nonzero in the absence of the singlet field. Finally, with $\bar{W}_i$, $i = 1, \ldots 23$ and $\bar{D}_i$, $i = 1, \ldots 3$ we define the extra octet and 27-plet operators that are nonzero in the presence of a dynamical singlet field.

The divergent octet W_i operators are defined as follows (the factor k is factorized everywhere):

$$
\begin{array}{rlrl}
W_1 & \langle \Delta u^2 u^2 \rangle \,, & W_{16} & i\langle \Delta \{ f_-^{\mu\nu}, u_\mu u_\nu \} \rangle \,, \\
W_2 & \langle \Delta u_\mu u^2 u^\mu \rangle \,, & W_{18} & \langle \Delta (f_+^2 - f_-^2) \rangle \,, \\
W_4 & \langle \Delta u_\mu \rangle \langle u^2 u^\mu \rangle \,, & W_{19} & i\langle \hat{\nabla}_\mu \Delta [u^\mu, u^2] \rangle \,, \\
W_5 & \langle \Delta \{ \chi_+, u^2 \} \rangle \,, & W_{20} & \langle \hat{\nabla}_\mu \Delta \{ w^{\mu\nu}, u_\nu \} \rangle \,, \\
W_6 & \langle \Delta u_\mu \rangle \langle \chi_+ u^\mu \rangle \,, & W_{21} & i\langle \hat{\nabla}_\mu \Delta [\chi_+, u^\mu] \rangle \,, \\
W_7 & \langle \Delta \chi_+ \rangle \langle u^2 \rangle \,, & W_{22} & \langle \hat{\nabla}_\mu \Delta d^\mu \chi_+ \rangle \,, \\
W_8 & \langle \Delta u^2 \rangle \langle \chi_+ \rangle \,, & W_{25} & \langle \hat{\nabla}_\mu \Delta \{ f_+^{\mu\nu}, u_\nu \} \rangle \,, \\
W_9 & \langle \Delta [\chi_-, u^2] \rangle \,, & W_{26} & \langle \hat{\nabla}_\mu \Delta \{ f_-^{\mu\nu}, u_\nu \} \rangle \,, \\
W_{10} & \langle \Delta \chi_+^2 \rangle \,, & W_{27} & \langle \Delta (2 f_+^2 - \{ f_+, f_- \}) \rangle \,, \\
W_{11} & \langle \Delta \chi_+ \rangle \langle \chi_+ \rangle \,, & W_{36} & \langle \Delta ([\chi_+, \chi_-] + \chi_+^2 - \chi_-^2) \rangle \,, \\
W_{12} & \langle \Delta \chi_-^2 \rangle \,, & W_{37} & \langle \Delta (f_+ + f_-)^2 \rangle \,, \\
W_{14} & i\langle \Delta \{ f_+^{\mu\nu}, u_\mu u_\nu \} \rangle \,, & W_{38} & \langle \Delta u_\mu \chi_+ u^\mu \rangle \,. \\
W_{15} & i\langle \Delta u_\mu f_+^{\mu\nu} u_\nu \rangle \,, & &
\end{array}
\tag{A.1}
$$

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The $\overline{W}_i$ octet operators are (the factor k is factorized everywhere):

$$\begin{array}{llll}
\bar{W}_1 & \langle \Delta\{u_\mu, u^2\}\rangle\langle u^\mu\rangle\,, & \bar{W}_{13} & \langle \hat{\nabla}_\mu \Delta w^{\mu\nu}\rangle\langle u_\nu\rangle\,, \\
\bar{W}_2 & \langle \Delta u_\mu u_\nu u^\mu\rangle\langle u^\nu\rangle\,, & \bar{W}_{14} & \langle \hat{\nabla}_\mu \Delta u_\nu\rangle\langle f_+^{\mu\nu}\rangle\,, \\
\bar{W}_3 & \langle \Delta\{\chi_+, u_\mu\}\rangle\langle u^\mu\rangle\,, & \bar{W}_{15} & \langle \hat{\nabla}_\mu \Delta f_+^{\mu\nu}\rangle\langle u_\nu\rangle \\
\bar{W}_4 & \langle \Delta[\chi_-, u_\mu]\rangle\langle u^\mu\rangle\,, & \bar{W}_{16} & \langle \hat{\nabla}_\mu \Delta u_\nu\rangle\langle f_-^{\mu\nu}\rangle\,, \\
\bar{W}_5 & i\langle \Delta[f_+^{\mu\nu}, u_\mu]\rangle\,, \langle u_\nu\rangle & \bar{W}_{17} & \langle \hat{\nabla}_\mu \Delta f_-^{\mu\nu}\rangle\langle u_\nu\rangle\,, \\
\bar{W}_6 & \langle \Delta u_\nu\rangle\langle f_+^{\mu\nu}\rangle\,, & \bar{W}_{18} & \langle \Delta u_\mu\rangle\langle u^2\rangle\langle u^\mu\rangle\,, \\
\bar{W}_7 & i\langle \Delta[u_\mu, u_\nu]\rangle\langle f_+^{\mu\nu}\rangle\,, & \bar{W}_{19} & \langle \Delta u_\mu\rangle\langle u^\mu u^\nu\rangle\langle u^\nu\rangle\,, \\
\bar{W}_8 & i\langle \Delta[u_\mu, u_\nu]\rangle\langle f_-^{\mu\nu}\rangle\,, & \bar{W}_{20} & \langle \Delta u^2\rangle\langle u_\mu\rangle\langle u^\mu\rangle\,, \\
\bar{W}_9 & \langle \Delta f_{\mu\nu+}\rangle\langle f_-^{\mu\nu}\rangle\,, & \bar{W}_{21} & \langle \Delta\{u_\mu, u_\nu\}\rangle\langle u^\mu\rangle\langle u^\nu\rangle\,, \\
\bar{W}_{10} & \langle \Delta f_{\mu\nu-}\rangle\langle f_+^{\mu\nu}\rangle\,, & \bar{W}_{22} & \langle \Delta \chi_+\rangle\langle u_\mu\rangle\langle u^\mu\rangle\,, \\
\bar{W}_{11} & i\langle \hat{\nabla}_\mu \Delta[u^\mu, u^\nu]\rangle\,, \langle u_\nu\rangle & \bar{W}_{23} & \langle \Delta u_\mu\rangle\langle u^\mu\rangle\langle \chi_+\rangle\,. \\
\bar{W}_{12} & \langle \hat{\nabla}_\mu \Delta u_\nu\rangle\langle w^{\mu\nu}\,,\rangle & &
\end{array} \tag{A.2}$$

The 27-plet D_i operators are given by (the factor q is factorized everywhere):

$$\begin{array}{llll}
D_1 & \langle \Delta\{u_\mu, u^2\}\rangle\langle \Delta u^\mu\rangle\,, & D_{13} & i\langle \Delta[f_+^{\mu\nu}, u_\mu]\rangle\langle \Delta u_\nu\rangle\,, \\
D_2 & \langle \Delta u_\mu u_\nu u^\mu\rangle\langle \Delta u^\nu\rangle\,, & D_{14} & i\langle \Delta f_+^{\mu\nu}\rangle\langle \Delta[u_\mu, u_\nu]\rangle\,, \\
D_3 & \langle \Delta\{u_\mu, u_\nu\}\rangle\langle \Delta\{u^\mu, u^\nu\}\rangle\,, & D_{15} & i\langle \Delta[u_\mu, u_\nu]\rangle\langle \Delta f_-^{\mu\nu}\rangle\,, \\
D_4 & \langle \Delta[u_\mu, u_\nu]\rangle\langle \Delta[u^\mu, u^\nu]\rangle\,, & D_{16} & \langle \Delta f_+^{\mu\nu}\rangle\langle \Delta f_{\mu\nu-}\rangle\,, \\
D_5 & \langle \Delta u^2\rangle\langle \Delta u^2\rangle\,, & D_{17} & i\langle \hat{\nabla}_\mu \Delta[u^\mu, u^\nu]\rangle\langle \Delta u_\nu\rangle\,, \\
D_6 & \langle \Delta u_\mu\rangle\langle \Delta u^\mu\rangle\langle u^2\rangle\,, & D_{18} & i\langle \hat{\nabla}_\mu \Delta u_\nu\rangle\langle \Delta[u^\mu, u^\nu]\rangle\,, \\
D_7 & \langle \Delta u_\mu\rangle\langle \Delta u_\nu\rangle\langle u^\mu u^\nu\rangle\,, & D_{19} & \langle \Delta w^{\mu\nu}\rangle\langle \hat{\nabla}_\mu \Delta u_\nu\rangle\,, \\
D_8 & \langle \Delta\{\chi_+, u_\mu\}\rangle\langle \Delta u^\mu\rangle\,, & D_{20} & \langle \hat{\nabla}_\mu \Delta w^{\mu\nu}\rangle\langle \Delta u_\nu\rangle\,, \\
D_9 & \langle \Delta u^2\rangle\langle \Delta \chi_+\rangle\,, & D_{21} & \langle \Delta f_+^{\mu\nu}\rangle\langle \hat{\nabla}_\mu \Delta u_\nu\rangle\,, \\
D_{10} & \langle \Delta u_\mu\rangle\langle \Delta u^\mu\rangle\langle \chi_+\rangle\,, & D_{22} & \langle \hat{\nabla}_\mu \Delta f_+^{\mu\nu}\rangle\langle \Delta u_\nu\rangle\,, \\
D_{11} & \langle \Delta[\chi_-, u_\mu]\rangle\langle \Delta u^\mu\rangle\,, & D_{23} & \langle \Delta f_-^{\mu\nu}\rangle\langle \hat{\nabla}_\mu \Delta u_\nu\rangle\,, \\
D_{12} & \langle \Delta \chi_+\rangle\langle \Delta \chi_+\rangle\,, & D_{24} & \langle \hat{\nabla}_\mu \Delta f_-^{\mu\nu}\rangle\langle \Delta u_\nu\rangle\,.
\end{array} \tag{A.3}$$

The 27-plet $\bar{D}_i$ operators are given by (the factor q is factorized everywhere):

$$\begin{array}{ll}
\bar{D}_1 & \langle \Delta u^2\rangle\langle \Delta u^\mu\rangle\langle u_\mu\rangle\,, \\
\bar{D}_2 & \langle \Delta\{u^\mu, u^\nu\}\rangle\langle \Delta u_\mu\rangle\langle u_\nu\rangle\,, \\
\bar{D}_3 & \langle \Delta u^\mu\rangle\langle \Delta \chi_+\rangle\langle u_\mu\,,\rangle\,.
\end{array} \tag{A.4}$$

The projection operator Δ_{ij} is defined as $\Delta_{ij} = u\lambda_{ij}u^\dagger$, with $(\lambda_{ij})_{ab} = \delta_{ia}\delta_{jb}$. The building blocks used in the definition of the counterterms are as follows

$$\begin{aligned}
u_\mu &= iu^\dagger D_\mu U u^\dagger = u_\mu^\dagger\,, \\
\chi_\pm &= u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \\
f_\pm^{\mu\nu} &= u f_L^{\mu\nu} u^\dagger \pm u^\dagger f_R^{\mu\nu} u\,, \\
w_{\mu\nu} &= d_\mu u_\nu + d_\nu u_\mu\,.
\end{aligned} \tag{A.5}$$

Also, the relations $f_-^{\mu\nu} = d^\nu u^\mu - d^\mu u^\nu$ and $f_+^{\mu\nu} = 2i\Gamma^{\mu\nu} - i/2[u^\mu, u^\nu]$ are useful. We defined as in ref. [5] the covariant derivative acting on Δ as $\hat{\nabla}_\mu \Delta = d_\mu \Delta + \frac{i}{2}[u_\mu, \Delta]$. The covariant derivative d_μ is the usual one acting on the fields in eq. (A.5). It is defined as

$$\begin{aligned} d_\mu O &= \partial_\mu O + [\Gamma_\mu, O]\,, \\ \Gamma_\mu &= \frac{1}{2}\left([u^\dagger, \partial_\mu u] - iu^\dagger r_\mu u - iul_\mu u^\dagger\right)\,. \end{aligned} \tag{A.6}$$

The equation of motion relates the field χ_- to the covariant derivative of the u_μ field

$$w_{\mu\mu} - \frac{1}{N}\langle w_{\mu\mu}\rangle \mathbf{1} = i\left(\chi_- - \frac{1}{N}\langle \chi_-\rangle \mathbf{1}\right)\,. \tag{A.7}$$

B. Integral over the singlet fields

In this Appendix we give the explicit expressions of the single terms that contribute to the ultraviolet divergent part in the perturbative expansion of the logarithm of the determinant of the singlet operator as defined in eq. (3.40). The result for each term reads as follows

$$\begin{aligned} \frac{i}{2}\mathrm{Tr} D_X^{0-1} A_w = &-\frac{1}{(4\pi)^2(d-4)}\int dx\, k\left\{\frac{1}{2N}(g_8' - g_8)\left(1 - \frac{N}{3}\alpha\right) M^2\langle\Delta\chi_+\rangle - \right. \\ &\left. -\frac{\bar{g}_8}{4}M^2\langle\Delta\chi_+\rangle + \frac{1}{6}(g_8' - g_8)(m_0^2 - \alpha M^2)\langle\Delta\chi_+\rangle\right\} + O(G_F^2)\,, \end{aligned} \tag{B.1}$$

$$\begin{aligned} -\frac{i}{2}\mathrm{Tr} D_X^{0-1} A_s D_X^{0-1} A_w = &-\frac{1}{(4\pi)^2(d-4)}\int dx\, k\left\{\frac{\alpha^2}{36}(g_8' - g_8)\langle\Delta\chi_+\rangle\langle\hat{\chi}_+\rangle + \right. \\ &+\frac{1}{4N^2}(g_8' - g_8)\left(1 - \frac{2}{3}N\alpha\right)\langle\Delta\chi_+\rangle\langle\hat{\chi}_+\rangle \\ &-\frac{\bar{g}_8}{8N}\left(1 - \frac{N}{3}\alpha\right)\langle\Delta\chi_+\rangle\langle\hat{\chi}_+\rangle - -\frac{1}{2}\langle\hat{\chi}_+\rangle\Big(\tilde{v}_8 k\langle\Delta u^2\rangle \\ &+\tilde{v}_5 k\langle\Delta\chi_+\rangle + \tilde{v}_{27} q\langle\Delta u_\mu\rangle\langle\Delta u^\mu\rangle + \tilde{v}_0 k\langle\Delta u_\mu\rangle\langle u^\mu\rangle\Big) - \\ &\left. -\frac{1}{2}\left(g_8' - g_8\right)\langle\Delta\chi_+\rangle\left(v_1\langle u^2\rangle + v_2\langle\hat{\chi}_+\rangle\right)\right\} + O(G_F^2)\,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} -\frac{i}{2}\mathrm{Tr}\left(D_X^{0-1} B_s^T \bar{D}^{-1} B_w + D_X^{0-1} B_w^T \bar{D}^{-1} B_s\right) = &-\frac{1}{(4\pi)^2(d-4)} \times \\ \times\int dx\Big\{-\frac{g_8}{6}\left(m_0^2 - \alpha M^2\right) k\langle\Delta\chi_+\rangle + g_8 k\left(1 - \frac{N}{3}\alpha\right)\Big[-\frac{i}{4N}\Big\langle\Big[d_\mu + \\ +\frac{i}{2}u_\mu, [\Delta, u^\mu]\Big]\chi_+\Big\rangle - \frac{1}{2N}\langle d^2\Delta\chi_+\rangle - \frac{1}{2N}\langle\Delta\chi_+^2\rangle + \frac{1}{2N^2}\langle\Delta\chi_+\rangle\langle\chi_+\rangle\Big] + \end{aligned}$$

$$
\begin{aligned}
&+g_8' k\left(1-\frac{N}{3}\alpha\right)\left[\frac{1}{2N}\langle\Delta\chi_+^2\rangle-\frac{1}{2N^2}\langle\Delta\chi_+\rangle\langle\chi_+\rangle+\frac{1}{4N}\langle\Delta[\chi_+,\chi_-]\rangle\right]+\\
&-\frac{1}{4N}g_{27}\left(1-\frac{N}{3}\alpha\right)q\langle\Delta\chi_+\rangle\langle\Delta\chi_+\rangle+\frac{1}{2}g_8k\langle\Delta\chi_+\rangle\Big(v_1\langle u^2\rangle+v_2\langle\hat\chi_+\rangle\Big)+\\
&+\bar g_8 k\left[\frac{\alpha}{24}\langle\Delta\chi_+\rangle\langle\hat\chi_+\rangle-\frac{i}{8}\langle[d_\mu+\frac{i}{2}u_\mu,[\Delta,u^\mu]]\chi_+\rangle-\frac{1}{4}\langle d^2\Delta\chi_+\rangle-\frac{1}{8}\langle\Delta\chi_+^2\rangle\right]-\\
&-g_8k\frac{\alpha}{12}\left[\langle\Delta u_\mu\chi_+u^\mu\rangle-\frac{1}{2}\langle\Delta\{\chi_+,u^2\}\rangle-\langle\Delta\chi_+^2\rangle+\frac{1}{N}\langle\Delta\chi_+\rangle\langle\chi_+\rangle\right]\Bigg\}+O(G_F^2)\,,
\end{aligned}
\tag{B.3}
$$

where $\bar D^{-1}$ is expanded up to order G_0^2 when the term proportional to $\alpha\Box$ inside B_w is considered.

$$
\begin{aligned}
\frac{i}{2}\mathrm{Tr}\left[D_X^{0\ -1}A_sD_X^{0\ -1}\Big(B_s^T\overline{D}^{-1}B_w+B_w^T\overline{D}^{-1}B_s\Big)\right]&=-\frac{1}{(4\pi)^2(d-4)}\times\\
&\times\int dx\, g_8\frac{\alpha}{12N}\langle\Delta\chi_+\rangle\langle\hat\chi_+\rangle+O(G_F^2)\,.
\end{aligned}
\tag{B.4}
$$

In eq. (B.5) only one term gives contribution to the ultraviolet divergent part of the singlet determinant. We used $u^2\equiv u_\mu u^\mu$ and $\langle\hat\chi_+\rangle=\langle\chi_+\rangle-2NM^2$. In the derivation of the complete result in terms of the operators listed in Appendix A some relations are useful

$$
\begin{aligned}
i\left\langle\left[d_\mu+\frac{i}{2}u_\mu,[\Delta,u^\mu]\right]\chi_+\right\rangle&=-i\left\langle\hat\nabla_\mu\Delta[\chi_+,u^\mu]\right\rangle+\frac{1}{2}\langle\Delta[\chi_+,\chi_-]\rangle\,,\\
\langle d^2\Delta\chi_+\rangle&=-\left\langle\hat\nabla_\mu\Delta d^\mu\chi_+\right\rangle+\frac{1}{4}\langle\Delta[\chi_+,\chi_-]\rangle-\frac{i}{2}\left\langle\hat\nabla_\mu\Delta[\chi_+,u^\mu]\right\rangle-\\
&\quad-\frac{1}{4}\langle\Delta\{\chi_+,u^2\}\rangle+\frac{1}{2}\langle\Delta u_\mu\chi_+u^\mu\rangle\,,
\end{aligned}
\tag{B.5}
$$

where $\hat\nabla_\mu\Delta=d_\mu\Delta+\frac{i}{2}[u_\mu,\Delta]$ and the building blocks are defined in eq. (A.5).

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